

NONLINEAR ELASTIC FREE ENERGIES AND GRADIENT YOUNG-GIBBS MEASURES

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ABSTRACT. We investigate, in a fairly general setting, the limit of large volume equilibrium Gibbs measures for elasticity type Hamiltonians with clamped boundary conditions. The existence of a quasiconvex free energy, forming the large deviations rate functional, is shown using a new interpolation lemma for partition functions. The local behaviour of the Gibbs measures can be parametrized by Young measures on the space of gradient Gibbs measures. In view of unboundedness of the state space, the crucial tool here is an exponential tightness estimate that holds for a vast class of potentials and the construction of suitable compact sets of gradient Gibbs measures.

1. SETTING AND RESULTS

The aim of the paper is to derive, in a mathematically rigorous way, macroscopic elasticity with variational principles formulated in terms of nonlinear elastic free energy from equilibrium statistical mechanics with gradient Gibbs measure on the space of displacements of individual atoms, based on a microscopic Hamiltonian.

We begin with the microscopic description. In general, we consider the space of microscopic configurations $X : \mathbb{Z}^d \rightarrow \mathbb{R}^m$. This includes the case of elasticity where we actually have $m = d$ with $X(i)$ denoting the vector of displacement of the atom labelled by i as well as the case of random interface with $m = 1$ and $X(i)$ denoting the height of interface above the lattice site i .

For any fixed $Y : \mathbb{Z}^d \rightarrow \mathbb{R}^m$ and any finite $\Lambda \subset \mathbb{Z}^d$, the Gibbs measure $\mu_{\Lambda,Y}(dX)$ on $(\mathbb{R}^m)^\Lambda$ under the boundary conditions Y is defined in terms of a Hamiltonian H with a finite range interaction U . Namely, let a finite $A \subset \mathbb{Z}^d$ and a function $U : (\mathbb{R}^m)^A \rightarrow \mathbb{R}$ be given. We use $R_0 = \text{diam} A$ to denote the range of potential U . We also assume that U is invariant under rigid motions (i.e. $U(\mathbf{R}(\tau_a X)) = U(X)$ for any $X \in (\mathbb{R}^m)^A$ and any $\mathbf{R} \in SO(m)$, $a \in \mathbb{R}^m$, with $\mathbf{R}(\tau_a X)(i) = \mathbf{R}(X(i) + a)$). In addition, suitable growth conditions on U will be specified later and, for simplicity (and without loss of generality), we suppose that $\{0, \pm e_1, \dots, \pm e_d\} \subset A$, where

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$\mathbf{e}_1, \dots, \mathbf{e}_d$ are the unit vectors in coordinate directions. Using X_A to denote the restriction of X to A for any $X : \mathbb{Z}^d \rightarrow \mathbb{R}^m$ and any $A \subset \mathbb{Z}^d$, we introduce the Hamiltonian

$$H_\Lambda(X) = \sum_{j \in \mathbb{Z}^d : \tau_j(A) \subset \Lambda} U(X_{\tau_j(A)}) \quad (1.1)$$

with $\tau_j(A) = A + j = \{i : i - j \in A\}$. The corresponding Gibbs measure on $(\mathbb{R}^m)^\Lambda$ (equipped with the corresponding Borel σ -algebra) is defined as

$$\mu_{\Lambda, Y}(dX) = \frac{\exp\{-\beta H_\Lambda(X)\}}{Z_{\Lambda, Y}} \mathbb{1}_{\Lambda, Y}(X) \prod_{i \in \Lambda} dX(i) \quad (1.2)$$

with

$$Z_{\Lambda, Y} = \int_{(\mathbb{R}^m)^\Lambda} \exp\{-\beta H_\Lambda(X)\} \mathbb{1}_{\Lambda, Y}(X) \prod_{i \in \Lambda} dX(i). \quad (1.3)$$

Here, we introduce boundary conditions by considering a fixed configuration Y in the boundary layer

$$S_{R_0}(\Lambda) = \{i \in \Lambda : \text{dist}(i, \mathbb{Z}^d \setminus \Lambda) \leq R_0\} \quad (1.4)$$

and restricting the configurations X to the set

$$\{X \in (\mathbb{R}^m)^\Lambda : |X(i) - Y(i)| < 1 \text{ for all } i \in S_{R_0}(\Lambda)\} \quad (1.5)$$

with the indicator $\mathbb{1}_{\Lambda, Y}(X)$. In the following we consider the inverse temperature β to be incorporated in the Hamiltonian and skip it from the notation.

In the standard setting of elasticity theory, we are interested in the macroscopic equilibrium configuration in an open set $\Omega \subset \mathbb{R}^d$ under fixed boundary conditions $u : \partial\Omega \rightarrow \mathbb{R}^m$. To link this with the microscopic description, we superimpose a finite lattice Ω_ε over Ω . Namely, for any $\varepsilon \in (0, 1)$, let

$$\Omega_\varepsilon = \frac{1}{\varepsilon}(\varepsilon\mathbb{Z}^d \cap \Omega) \equiv \mathbb{Z}^d \cap \frac{1}{\varepsilon}\Omega. \quad (1.6)$$

Naturally, $\frac{1}{\varepsilon}\Omega$ and $\varepsilon\mathbb{Z}^d$ denotes the rescaling of Ω and \mathbb{Z}^d by $\frac{1}{\varepsilon}$ and ε , respectively. We will assume certain regularity of the boundary $\partial\Omega$ of the domain Ω . Namely, using $\partial_\varrho\Omega$ to denote the intersection of the ϱ -neighbourhood of the boundary $\partial\Omega$ with Ω ,

$$\partial_\varrho\Omega = \{x \in \Omega : \text{dist}(x, \partial\Omega) < \varrho\}, \quad \varrho > 0, \quad (1.7)$$

we assume that Ω is a domain with Lipschitz boundary that allows to check the following condition:

(A $_\partial$) *There exist constants ρ_0 , ε_0 , and C_∂ such that, for any $\varrho \leq \rho_0$ and $\varepsilon \leq \varepsilon_0$, the number of points in the strip $S_{\rho/\varepsilon} = (\partial_\varrho\Omega)_\varepsilon = \{i \in \Omega_\varepsilon \mid i\varepsilon \in \partial_\varrho\Omega\}$ is bounded as $|S_{\rho/\varepsilon}| \leq \varepsilon^{-d} C_\partial |\partial\Omega| \varrho$.*

Further, for any $u \in L_{1,\text{loc}}(\mathbb{R}^d, \mathbb{R}^m)$, let $X_{u,\varepsilon} : \mathbb{Z}^d \rightarrow \mathbb{R}^m$ be defined by

$$X_{u,\varepsilon}(i) = \frac{1}{\varepsilon} \oint_{\varepsilon i + Q(\varepsilon)} u(y) dy \quad (1.8)$$

for any $i \in \mathbb{Z}^d$. Here, $Q(\varepsilon) = [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]^d$ and \oint denotes the mean value. On the other hand, let

$$\Pi_\varepsilon : (\mathbb{R}^m)^{\mathbb{Z}^d}_0 \rightarrow W^{1,p}(\mathbb{R}^d) \quad (1.9)$$

be a canonical interpolation $X \rightarrow v$ such that $v(\varepsilon i) = \varepsilon X(i)$ for any $i \in \mathbb{Z}^d$. Here $(\mathbb{R}^m)^{\mathbb{Z}^d}_0$ is the set of functions $X : \mathbb{Z}^d \rightarrow \mathbb{R}^m$ with finite support. To fix the ideas, we can consider a triangulation of \mathbb{Z}^d into simplices with vertices in $\varepsilon \mathbb{Z}^d$ and choose v on each simplex as the linear interpolation of the values $\varepsilon X(i)$ on the vertices εi .

Our main task is to study the asymptotic behaviour (with $\varepsilon \rightarrow 0$) of the measure $\mu_{\Omega_\varepsilon, X_{u,\varepsilon}}(dX)$ in terms of minimizers v of the functional $\int_\Omega W(\nabla v(x)) dx$ with the boundary condition $v = u$ on $\partial\Omega$. It turns out that the free energy $W(L)$ featuring in the above integral is defined, for any affine function $L : \mathbb{R}^d \rightarrow \mathbb{R}^m$, by the limit

$$W(L) = -\lim_{\varepsilon \rightarrow 0} \varepsilon^d |\Omega|^{-1} \log Z_{\Omega_\varepsilon, L}, \quad (1.10)$$

where $Z_{\Omega_\varepsilon, L}$ is a shorthand for $Z_{\Omega_\varepsilon, X_{L,\varepsilon}}$ reflecting the fact that, with an affine function L , we actually have $X_{L,\varepsilon}(i) = L(i)$ with the condition $|X - L|_{S_{R_0}(\Omega_\varepsilon), \infty} \leq 1$ reading $|\Pi_\varepsilon(X) - L|_{S_{R_0/\varepsilon}(\Omega), \infty} \leq \varepsilon$. For the existence of the limit, see Proposition 1.2 below.

Using $\nabla X(i) = (\nabla_1 X(i), \dots, \nabla_d X(i))$ for the discrete gradient, $\nabla_k X(i) = X(i + e_k) - X(i)$, $k = 1, \dots, d$, and defining $|\nabla X(i)|^p = \sum_{k=1}^d |\nabla_k X(i)|^p$, our main assumptions are the following basic restrictions on the growth of the potential U (part of the lower bound is the boundedness from below that can be stated, without loss of generality, as an assumption of non-negativity):

(A1) *There exist constants $p > 0$ and $c \in (0, \infty)$ such that*

$$U(X_A) \geq c |\nabla X(0)|^p$$

for any $X \in (\mathbb{R}^m)^{\mathbb{Z}^d}$.

(A2) *There exist constants $r > 1$ and $C \in (1, \infty)$ such that*

$$U(sX_A + (1-s)Y_A + Z_A) \leq C(1 + U(X_A) + U(Y_A) + \sum_{i \in A} |Z(i)|^r)$$

for any $s \in [0, 1]$ and any $X, Y, Z \in (\mathbb{R}^m)^{\mathbb{Z}^d}$.

Increasing possibly the constant C to incorporate the term $U(0)$ from (A2) applied with $s = 1$ and $Y = 0$, we have a particular useful implication of (A2) in the form

$$U(X_A) \leq C(1 + U(Z_A) + \sum_{i \in A} |X(i) - Z(i)|^r). \quad (1.11)$$

Remark 1.1. *In view of the invariance of the function U under rigid motions, it actually depends only on gradients $\nabla X(i)$, $i \in A$. With the help of discrete Poincaré inequality, the condition (A2) implies*

$$U(X_A) \leq C \left(\sum_{i \in Q(R_0)} |\nabla X(i)|^r + 1 \right), \quad (1.12)$$

with a suitable constant C (again not necessarily the same as that in (A2)) and $Q(R_0) \subset \mathbb{Z}^d$ denoting a cube (of side $R_0 = \text{diam} A$) containing A .

1.1. Free energy. A prerequisite to our main statements is the existence of the free energy as a function of the affine deformation L and its continuity and quasiconvexity.

Proposition 1.2 (Existence of the free energy).

Suppose that (A2) holds with $r \geq 1$. Then, for any affine $L : \mathbb{R}^d \rightarrow \mathbb{R}^m$, the limit

$$W(L) = - \lim_{\varepsilon \rightarrow 0} \varepsilon^d |\Omega|^{-1} \log Z_{\Omega_\varepsilon, L} \quad (1.13)$$

exists and does not depend on Ω .

Remark 1.3. *Instead of the condition (A2), it is enough here to assume that $U(X_A)$ is bounded by a fixed constant for any X such that $|X(i) - L(i)| \leq 1$ for all $i \in A$.*

Proof. The existence of the limit and its independence on Ω follows easily by the standard methods with the help of an approximative subadditivity (of $-\log Z_{\Lambda, L}$): if $\Lambda \subset \mathbb{Z}^d$ is finite and Λ_1 and Λ_2 are its disjoint subsets, $\Lambda_1 \cup \Lambda_2 = \Lambda$, then

$$\log Z_{\Lambda, L} \geq \log Z_{\Lambda_1, L} + \log Z_{\Lambda_2, L} - B(L) |S(\Lambda_1, \Lambda_2)|, \quad (1.14)$$

where $B(L) = C(1 + C + (1 + Cd\|L\|^r)R_0^d)$ is a constant and

$$S(\Lambda_1, \Lambda_2) = \Lambda \cap (\Lambda_1)_{R_0} \cap (\Lambda_2)_{R_0}, \quad (1.15)$$

where $(\Lambda_k)_{R_0}$, $k = 1, 2$, is the R_0 -neighbourhood of Λ_k . Indeed, using the assumption (A2) (resp. its implication (1.11)), we have

$$U(X_{\tau_j(A)}) \leq C(1 + U(L_{\tau_j(A)}) + R_0^d), \quad (1.16)$$

for all j such that $\tau_j(A) \subset \Lambda$ and in the same time $\tau_j(A) \cap \Lambda_1 \neq \emptyset$ as well as $\tau_j(A) \cap \Lambda_2 \neq \emptyset$ (which implies $\tau_j(A) \subset S(\Lambda_1, \Lambda_2)$). Hence,

$$H_\Lambda(X) \leq H_{\Lambda_1}(X) + H_{\Lambda_2}(X) + B(L) |S(\Lambda_1, \Lambda_2)| \quad (1.17)$$

for any X satisfying $\mathbb{1}_{\Lambda_1, L}(X) \mathbb{1}_{\Lambda_2, L}(X) = 1$. Thus, restricting first the range of integration in the definition of $Z_{\Lambda, L}$ by inserting the indicator $\mathbb{1}_{\Lambda_1, L}(X) \mathbb{1}_{\Lambda_2, L}(X)$ (notice that $\mathbb{1}_{\Lambda_1, L}(X) \mathbb{1}_{\Lambda_2, L}(X) \leq \mathbb{1}_{\Lambda, L}(X)$) and using then the inequality (1.17), we get the claim. \square

Proposition 1.4 (Quasiconvexity of the free energy).

Assume that U satisfies the assumptions (A1) and (A2) with $r, p \geq 1$, $\frac{1}{r} > \frac{1}{p} - \frac{1}{d}$. The free energy $W(L)$ is continuous with r growth,

$$|W(L)| \leq \overline{C}(1 + \|L\|^r), \quad (1.18)$$

it is quasiconvex, and, as a consequence, $\int_{\Omega} W(\nabla v(x)) dx$ is a weakly lower semicontinuous functional on $W^{1,r}(\Omega)$.

Remark 1.5. Notice that, in general, for $m \geq 2$, we should not expect that the free energy is convex. We provide an explicit class of examples in Section 2.6.

The proof of this and the remaining statements in this section is deferred to the next section as it hinges on the crucial Interpolation Lemma and Exponential Tightness Lemma to be stated there (and proven in the Appendix).

1.2. Large deviations. Before formulating the theorem whose consequence is the large deviations principle for the measure $\mu_{\Omega_\varepsilon, X_{u,\varepsilon}}(dX)$, we introduce several restricted partition functions. For any finite $\Lambda \subset \mathbb{Z}^d$ and any set $\mathcal{S} \subset (\mathbb{R}^m)^\Lambda$, we write

$$Z_\Lambda(\mathcal{S}) = \int_{\mathcal{S}} \exp\{-H_\Lambda(X)\} \prod_{i \in \Lambda} dX(i). \quad (1.19)$$

In particular, for any $Y \in (\mathbb{R}^m)^{\mathbb{Z}^d}$,

$$Z_{\Lambda,Y} = Z_\Lambda(\mathcal{N}_{\Lambda,R_0,\infty}(Y)), \quad (1.20)$$

where $\mathcal{N}_{\Lambda,R_0,\infty}(Y)$ is the set corresponding to the indicator $\mathbb{1}_{\Lambda,Y}$,

$$\mathcal{N}_{\Lambda,R_0,\infty}(Y) = \{X : \Lambda \rightarrow \mathbb{R}^m \mid |X(i) - L(i)| < 1 \text{ for all } i \in S_{R_0}(\Lambda)\}. \quad (1.21)$$

Now, for any $v \in L^r(\Omega)$, consider the neighbourhood

$$\mathcal{N}_{\Omega_\varepsilon,r}(v, \kappa) = \{X : \Omega_\varepsilon \rightarrow \mathbb{R}^m \mid \|\Pi_\varepsilon(X) - v\|_{L^r(\Omega)} < \kappa |\Omega|^{|\frac{1}{r} + \frac{1}{d}|}\} \subset (\mathbb{R}^m)^{\Omega_\varepsilon} \quad (1.22)$$

with the corresponding partition function

$$Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon,r}(v, \kappa)) = \int_{\mathcal{N}_{\Omega_\varepsilon,r}(v, \kappa)} \exp\{-H_{\Omega_\varepsilon}(X)\} \prod_{i \in \Omega_\varepsilon} dX(i). \quad (1.23)$$

Notice that the volume dependent factor $|\Omega|^{|\frac{1}{r} + \frac{1}{d}|}$ is chosen so that the set $\mathcal{N}_{\Omega_\varepsilon,r}(v, \kappa)$ does not change under the rescaling $\varepsilon \rightarrow \sigma\varepsilon$ and $\Omega \rightarrow \sigma\Omega$ with $v(x) \rightarrow \sigma v(\frac{x}{\sigma})$. It is easy to verify the following equivalences of the corresponding norms (uniformly in ε),

$$\frac{1}{2} \|\Pi_\varepsilon(X) - v\|_{L^r(\Omega)}^r \leq \varepsilon^{d+r} \sum_{i \in \Omega_\varepsilon} |X(i) - X_{v,\varepsilon}(i)|^r \leq 2 \|\Pi_\varepsilon(X) - v\|_{L^r(\Omega)}^r \quad (1.24)$$

and

$$\frac{1}{2} \|\nabla(\Pi_\varepsilon(X) - v)\|_{L^p(\Omega)}^p \leq \varepsilon^d \sum_{i \in \Omega_\varepsilon} |\nabla X(i) - \nabla X_{v,\varepsilon}(i)|^p \leq 2 \|\nabla(\Pi_\varepsilon(X) - v)\|_{L^p(\Omega)}^p. \quad (1.25)$$

Using χ_ε to denote the characteristic function of $Q(\varepsilon)$ and comparing piecewise linear and piecewise constant interpolation, we get

$$\|\Pi_\varepsilon(X) - \sum_i \varepsilon \chi_\varepsilon(\cdot - \varepsilon i) X(i)\|_{L^r(\Omega)}^r \leq 2 \varepsilon^{d+r} \sum_{i \in \Omega_\varepsilon} |\nabla X(i)|^\alpha |X(i)|^{r-\alpha} \quad (1.26)$$

for any $\alpha < 1$, which by the Sobolev estimates entails

$$\|\Pi_\varepsilon(X) - \sum_i \varepsilon \chi_\varepsilon(\cdot - \varepsilon i) X(i)\|_{L^r(\Omega)}^r \leq 2 \varepsilon^\alpha \|\Pi_\varepsilon(X)\|_{W^{1,p}}^r, \quad (1.27)$$

where $\alpha = \min(1, d + r - \frac{dr}{p})$.

In view of (1.24), the condition $\|\Pi_\varepsilon(X) - v\|_{L^r(\Omega)}^r < \kappa^r |\Omega|^{1+\frac{r}{d}}$ is, up to a change of κ multiplying it by a fixed factor, equivalent to $\sum_{i \in \Omega_\varepsilon} |X(i) - X_{v,\varepsilon}(i)|^r < \kappa^r |\Omega_\varepsilon|^{1+\frac{r}{d}}$. This suggests the notation

$$\overline{\mathcal{N}}_{\Lambda,r}(Z, \kappa) = \{X : \Lambda \rightarrow \mathbb{R}^m \mid \|X - Z\|_{\ell^r(\Lambda)} < \kappa |\Lambda|^{\frac{1}{r} + \frac{1}{d}}\} \quad (1.28)$$

for any $\Lambda \subset \mathbb{Z}^d$ and $Z \in (\mathbb{R}^m)^\Lambda$. As observed above, for $\Lambda = \Omega_\varepsilon$ and $Z = X_{v,\varepsilon}$, the sets $\mathcal{N}_{\Omega_\varepsilon,r}(v, \kappa)$ and $\overline{\mathcal{N}}_{\Omega_\varepsilon,r}(Z, \kappa)$ actually differ only by change of κ up to the factor 2, $\kappa \rightarrow 2\kappa$.

Theorem 1.6.

Assume that U satisfies the assumptions (A1) and (A2) with $r \geq p > 1$, $\frac{1}{r} > \frac{1}{p} - \frac{1}{d}$ and let $v \in W^{1,p}(\Omega)$. Further, let

$$F_{\kappa,\varepsilon}(v) = -\varepsilon^d |\Omega|^{-1} \log Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon,r}(v, \kappa)), \quad (1.29)$$

and

$$F_\kappa^+(v) = \limsup_{\varepsilon \rightarrow 0} F_{\kappa,\varepsilon}(v) \quad (1.30)$$

$$F_\kappa^-(v) = \liminf_{\varepsilon \rightarrow 0} F_{\kappa,\varepsilon}(v) \quad (1.31)$$

Then:

a) $\lim_{\kappa \rightarrow 0} F_\kappa^-(v) \geq \frac{1}{|\Omega|} \int_\Omega W(\nabla v(x)) dx$.

b) If $v \in W^{1,r}(\Omega)$ then $\lim_{\kappa \rightarrow 0} F_\kappa^+(v) \leq \frac{1}{|\Omega|} \int_\Omega W(\nabla v(x)) dx$.

As a consequence, we get the following large deviation behaviour for $\mu_{\Omega_\varepsilon, X_{u,\varepsilon}}(dX)$. For convenience, we actually extend the measure $\mu_{\Omega_\varepsilon, X_{u,\varepsilon}}(dX)$ defined on $(\mathbb{R}^m)^{\Omega_\varepsilon}$ to $(\mathbb{R}^m)^{\mathbb{Z}^d}$ by defining $\mu_{\varepsilon,u} = \mu_{\Omega_\varepsilon, X_{u,\varepsilon}} \times \prod_{i \in \mathbb{Z}^d \setminus \Omega_\varepsilon} \delta_{X_{u,\varepsilon}(i)}$ and adding the assumption that the function u is supported on a bounded set in \mathbb{R}^d .

Theorem 1.7 (Large deviation principle).

Assume that U satisfies the assumptions (A1) and (A2) with $r \geq p > 1$, $\frac{1}{r} > \frac{1}{p} - \frac{1}{d}$, and let $u \in W^{1,p}(\Omega)$. If $p = r$ or, more generally,

$$\int_{\Omega} W(\nabla v(x)) dx = \sup_{\delta > 0} \inf \left\{ \int_{\Omega} W(\nabla \bar{v}(x)) dx \mid \bar{v} \in W_0^{1,r}(\Omega) + u, \|\bar{v} - v\|_{L^r} < \delta \right\}, \quad (1.32)$$

for every $v \in W_0^{1,p}(\Omega) + u$, then the Gibbs measures $\mu_{\varepsilon,u}(dX)$ satisfies the large deviation principle with the rate ε^{-d} and the rate functional

$$I(v) = \int_{\Omega} W(\nabla v(x)) dx - \min_{\bar{v} \in W_0^{1,r}(\Omega) + u} \int_{\Omega} W(\nabla \bar{v}(x)) dx. \quad (1.33)$$

Namely:

a) For any $C \subset W_0^{1,p}(\Omega) + u$ closed in the weak topology, we have

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^d \log \mu_{\varepsilon,u}(\Pi_{\varepsilon}^{-1}(C)) \leq - \inf_{v \in C} I(v). \quad (1.34)$$

b) For any $O \subset W_0^{1,p}(\Omega) + u$ open in the weak topology, we have

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^d \log \mu_{\varepsilon,u}(\Pi_{\varepsilon}^{-1}(O)) \geq - \inf_{v \in O} I(v). \quad (1.35)$$

Remark 1.8. Similar statements hold for Neumann or periodic boundary conditions.

The existence of $v \in W_0^{1,p}(\Omega) + u$ such that there is a strict inequality in (1.32) is the so called Lavrentiev gap. Thus, our claim about LDP comes under the assumption of the absence of Lavrentiev gap.

Remark 1.9. Suppose that $F : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ is weakly continuous and has a growth strictly smaller than p . Then the rate functional corresponding to $H + F \circ \Pi_{\varepsilon}$ (i.e. the rate functional of the measure $\frac{e^{F(\Pi_{\varepsilon}(X))} \mu_{\varepsilon}(dX)}{\int e^{F(\Pi_{\varepsilon}(Y))} \mu_{\varepsilon}(dY)}$) is

$$\int_{\Omega} W(\nabla v(x)) dx + F(v) - \min_{\bar{v} \in W_0^{1,r}(\Omega) + u} \left[\int_{\Omega} W(\nabla \bar{v}(x)) dx + F(\bar{v}) \right]. \quad (1.36)$$

We note that large deviation principle has been discussed before (often under more restrictive conditions on the potential and in the scalar case $m = 1$). See, for example, [1] (the case of strictly convex potentials and $m = 1$) and [2] (potentials satisfying analog of our assumptions (A1) and (A2) with $r = p$ and with large deviation formulated in detail only for $m = 1$).

While the large deviation principle clarifies the role of minimizers of the functional I for the description of the “macroscopic” asymptotic behaviour of measures $\mu_{\varepsilon,u}$, our final claim, introducing the notion of “Young-Gibbs” measures, inspects the asymptotic “microscopic” behaviour of $\mu_{\varepsilon,u}$.

1.3. Gradient Young-Gibbs measures. Here, we will need the Gibbs specification $\mu_\Lambda(dX|Y)$ in the volume $\Lambda \subset \mathbb{Z}^d$ and with boundary condition Y . First, for any $X, Y \in (\mathbb{R}^m)^{\mathbb{Z}^d}$, we introduce the Hamiltonian

$$H_\Lambda(X|Y) = \sum_{j \in \mathbb{Z}^d: \tau_j(\Lambda) \cap \Lambda \neq \emptyset} U((X \vee Y)_{\tau_j(\Lambda)}), \text{ where } X \vee Y = \begin{cases} X & \text{on } \Lambda \\ Y & \text{otherwise.} \end{cases} \quad (1.37)$$

Then,

$$\mu_\Lambda(dX|Y) = \frac{\exp\{-H_\Lambda(X|Y)\}}{Z_\Lambda(Y)} \prod_{i \in \Lambda} dX(i) \prod_{i \in \Lambda^c} \delta_{Y(i)}(dX(i)) \quad (1.38)$$

with

$$Z_\Lambda(Y) = \int_{(\mathbb{R}^m)^\Lambda} \exp\{-H_\Lambda(X|Y)\} \prod_{i \in \Lambda} dX(i). \quad (1.39)$$

A gradient Gibbs measure (with potential U) is any probability measure μ on the space $S = (\mathbb{R}^m)^{\mathbb{Z}^d}/\mathbb{R}^m$ such that

$$\mu(f) = \int \mu_\Lambda(f|Y) \mu(dY) \quad (1.40)$$

for any finite $\Lambda \subset \mathbb{Z}^d$ and any measurable function f on S (i.e. a measurable function on $(\mathbb{R}^m)^{\mathbb{Z}^d}$ invariant under translations in \mathbb{R}^m). Notice that for such f , the function $Y \rightarrow \mu_\Lambda(f|Y)$ is also invariant under translations in \mathbb{R}^m and can thus be integrated with the probability measure μ on S . We use \mathcal{G} to denote the set of all gradient Gibbs measures and \mathcal{G}^p to denote the set of all $\mu \in \mathcal{G}$ with finite p th moment. Both are subsets of the set $\mathcal{P}(S)$ of probability measures on the space S .

Further, we introduce two measure spaces on sets of measures, both with the corresponding weak topology: the space $\mathcal{P}(\mathcal{G}^p)$ of probability measures on \mathcal{G}^p and the space $\mathcal{P}(W_0^{1,p}(\Omega) + u)$ of probability measures on $W_0^{1,p}(\Omega) + u$ (nonnegative functionals on bounded weakly continuous functions on $W_0^{1,p}(\Omega) + u$ of mass 1). A probability measure $\gamma \in \mathcal{P}(W_0^{1,p}(\Omega) + u)$ together with a map $\nu : \Omega \times (W_0^{1,p}(\Omega) + u) \rightarrow \mathcal{P}(\mathcal{G}^p) \subset BC(BC(S)^*)^*$ that is weakly measurable with respect to $\gamma \times \lambda$, where λ is the normalized Lebesgue measure on Ω , is called a *gradient Young-Gibbs measure*.

We will show that the measures $\mu_{\varepsilon,u}$ converge, in appropriate sense, to a gradient Young-Gibbs measure with a slope condition linking the slope $\nabla v(x)$ with the expectation $\int \mathbb{E}_\mu(\nabla X(0)) d\nu_{x,v}(\mu)$.

We formulate the convergence in terms of appropriate test functions. Namely, we consider the space \mathcal{X} of *test functions*

$$\varphi : (W_0^{1,p}(\Omega) + u) \times BC(S)_+^* \times \Omega \rightarrow \mathbb{R} \quad (1.41)$$

that are weakly continuous and for any δ fulfill the growth condition

$$|\varphi(v, \mu, x)| \leq \eta(x) \exp\{c_\varphi \|v\|_{1,p}^p\} \mu(\delta \sum_{i \in \Lambda} |\nabla X(i)|^p + C(\delta)) \quad (1.42)$$

with a fixed Λ , $\eta \in C_0(\Omega)$ and the constant c_φ depending only on φ .

Theorem 1.10 (Convergence to gradient Young-Gibbs measures).

Assume that the potential U satisfies the assumptions (A1) and (A2) with $r \geq p > 1$, $\frac{1}{r} > \frac{1}{p} - \frac{1}{d}$ and let $u \in W^{1,p}(\Omega)$. There exists a sequence $\varepsilon_n \rightarrow 0$ and a gradient Young-Gibbs measure ν with γ supported, if the functional I has no Lavrentiev gap, on minimisers of I in $W_0^{1,p}(\Omega) + u$ such that

$$\begin{aligned} \lim_{\Lambda \rightarrow \mathbb{Z}^d} \lim_{n \rightarrow \infty} \mu_{\varepsilon_n, u} \left(\int_{\Omega} \varphi(\Pi_{\varepsilon}(X), \mu_{\Lambda}(\cdot | \tau_{\lfloor x/\varepsilon \rfloor}(X)), x) dx \right) = \\ = \int_{W_0^{1,p}(\Omega) + u} \int_{\Omega} \nu_{x,v}(\varphi(v, \cdot, x)) d\gamma(v) dx \end{aligned} \quad (1.43)$$

for all $\varphi \in \mathcal{X}$ and

$$\int \mathbb{E}_{\mu}(\nabla X(0)) d\nu_{x,v}(\mu) = \nabla v(x) \quad (1.44)$$

for almost all x and almost all v (with respect to γ).

The idea of of gradient Young-Gibbs measures has its precursor in the non-stochastic case, where the Young measures and the methods of Γ -convergence were often used in the context of nonlinear elasticity (see [3] for a review). We were also inspired by the procedure of two-scale convergence that is used in homogenization. See, for example, [4]. Whenever there is a unique Gibbs measure corresponding to the affine mapping $L = \nabla v(x)$, the measure $\nu_{x,v}$ is actually a Dirac measure. For the scalar case, $m = 1$, the unicity of Gibbs measure corresponding to a fixed slope has been proven for a class of strictly convex potentials [5]. Notice, however, that even in the scalar case, this is not always the case as phase transitions may occur [6].

2. PROOFS

2.1. Exponential Tightness. Here, we first state two crucial Lemmas (with the proofs deferred to the Appendix) and then, using them, we prove the claims from Section 1. In the following, we consider m, d , and Ω to be fixed (often without explicitly mentioning the dependence of various constants on these parameters).

The first technical Lemma assures a needed tightness of finite volume Gibbs measures when conditioned on the neighbourhood $\mathcal{N}_{\Omega_{\varepsilon}, r}(v, \kappa)$. Let, for any $K \in (0, \infty)$, the set \mathcal{M}_K be defined by

$$\mathcal{M}_K = \{X : \Omega_{\varepsilon} \rightarrow \mathbb{R}^m | H_{\Omega_{\varepsilon}}(X) > K |\Omega_{\varepsilon}| \}. \quad (2.1)$$

Lemma 2.1 (Exponential Tightness).

Assume that U satisfies the assumption (A1). There exists a constant K_0 and, for any r , and κ , a constant $\varepsilon_0 = \varepsilon_0(r, \kappa)$ such that, for any $\varepsilon \leq \varepsilon_0$, $K \geq K_0$,

$u \in L_{1,loc}(\Omega, \mathbb{R}^m)$, and $v \in L^r(\Omega)$, we have

$$Z_{\Omega_\varepsilon}(\mathcal{M}_K \cap \mathcal{N}_{\Omega_\varepsilon, R_0, \infty}(X_{u, \varepsilon})) \leq e^{-\frac{1}{2}K|\Omega_\varepsilon|} D^{|\Omega_\varepsilon|} \quad (2.2)$$

and

$$Z_{\Omega_\varepsilon}(\mathcal{M}_K \cap \mathcal{N}_{\Omega_\varepsilon, r}(v, \kappa)) \leq e^{-\frac{1}{2}K|\Omega_\varepsilon|} D^{|\Omega_\varepsilon|} \quad (2.3)$$

with $D = 2(\frac{2}{c})^{\frac{m}{p}} c(p, m)$, where $c(p, m) = \int_{\mathbb{R}^m} \exp(-|\xi|^p) d\xi$.

Remark 2.2. We refer to the above claim as the exponential tightness since, under additional assumption (A2), it implies that

$$\begin{aligned} \mu_{\varepsilon, u}(\mathcal{M}_K) &= \frac{Z_{\Omega_\varepsilon}(\mathcal{M}_K \cap \mathcal{N}_{\Omega_\varepsilon, R_0, \infty}(X_{u, \varepsilon}))}{Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon, R_0, \infty}(X_{u, \varepsilon}))} \leq \\ &\leq e^{-\frac{1}{2}K|\Omega_\varepsilon|} (D/\omega(m))^{|\Omega_\varepsilon|} \exp(C(H_{\Omega_\varepsilon}(X_{u, \varepsilon}) + (1 + R_0^d)|\Omega_\varepsilon|)) \end{aligned} \quad (2.4)$$

with $\omega(m)$ denoting the volume of a unit ball in \mathbb{R}^m . Indeed, considering the $\ell^\infty(\Omega_\varepsilon)$ -neighbourhood

$$\mathcal{N}_{\Omega_\varepsilon, \infty}(Z) = \{X : \Omega_\varepsilon \rightarrow \mathbb{R}^m \mid |X(i) - Z(i)| < 1 \text{ for all } i \in \Omega_\varepsilon\}, \quad (2.5)$$

and using (A2) in the form (1.11), we get

$$Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon, R_0, \infty}(X_{u, \varepsilon})) \geq Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon, \infty}(X_{u, \varepsilon})) \geq \exp(-C(H_{\Omega_\varepsilon}(X_{u, \varepsilon}) + (1 + R_0^d)|\Omega_\varepsilon|)) \omega(m)^{|\Omega_\varepsilon|}. \quad (2.6)$$

Similarly, observing that for any $X \in \mathcal{N}_{\Omega_\varepsilon, \infty}(Z)$ and a sufficiently small ε , we have $|\Omega_\varepsilon|^{1/d} > 2/\kappa$ and thus

$$\sum_{i \in \Omega_\varepsilon} |X(i) - Z(i)|^r \leq |\Omega_\varepsilon| \leq (\frac{\kappa}{2})^r |\Omega_\varepsilon|^{1+\frac{r}{d}}, \quad (2.7)$$

implying $\mathcal{N}_{\Omega_\varepsilon, \infty}(Z) \subset \overline{\mathcal{N}}_{\Omega_\varepsilon, r}(Z, \kappa/2) \subset \mathcal{N}_{\Omega_\varepsilon, r}(v, \kappa)$, we get

$$\frac{Z_{\Omega_\varepsilon}(\mathcal{M}_K \cap \mathcal{N}_{\Omega_\varepsilon, r}(v, \kappa))}{Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon, r}(v, \kappa))} \leq e^{-\frac{1}{2}K|\Omega_\varepsilon|} (D/\omega(m))^{|\Omega_\varepsilon|} \exp(C(H_{\Omega_\varepsilon}(Z) + (1 + R_0^d)|\Omega_\varepsilon|)) \quad (2.8)$$

for any $Z \in \mathcal{N}_{\Omega_\varepsilon, r}(v, \kappa/2)$.

2.2. Interpolation. The crucial step in the proof of the Large Deviation statement is based on the possibility to approximate with partition functions on cells of a triangulation given in terms of L^r -neighbourhoods of linearizations of a minimiser of the rate functional. An important tool that will eventually allow to impose a boundary condition on each cell of the triangulation consists in switching between the corresponding partition function $Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon, r}(v, \kappa))$ and the version $Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon, r}(v, 2\kappa) \cap \mathcal{N}_{\Omega_\varepsilon, R_0, \infty}(Z))$ with an additional soft clamp $|X(i) - Z(i)| < 1$ enforced in the boundary strip of the width $R_0 > \text{diam}(A)$ with $Z \in \mathcal{N}_{\Omega_\varepsilon, r}(v, \kappa)$ arbitrarily chosen.

Fixing parameters $\eta > 0$ and $N \in \mathbb{N}$, we will slice the strip $\partial_{\eta/\varepsilon}\Omega_\varepsilon$ into strips of width $\frac{\eta}{\varepsilon N}$ that will provide a framework for the interpolation. Recalling the notation

$$F_{\kappa,\varepsilon}(v) = -\varepsilon^d |\Omega|^{-1} \log Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon,r}(v, \kappa)), \quad (2.9)$$

we have the following claim.

Lemma 2.3 (Interpolation).

Suppose that U satisfies the assumptions (A1) and (A2). There exist constants κ_0, b , and \mathcal{C} (depending on $c, C, R_0, |\Omega|, |\partial\Omega|, p$, and m) and a function $\varepsilon_0(\kappa, \eta, N)$ such that

$$\begin{aligned} Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon,r}(v, \kappa)) &\leq Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon,r}(v, 2\kappa) \cap \mathcal{N}_{\Omega_\varepsilon,R_0,\infty}(Z)) \times \\ &\times \exp\left\{\mathcal{C}\left(\left(\frac{b+F_{\kappa,\varepsilon}(v)}{N}\right) + \eta + \left(\frac{N\kappa}{\eta}\right)^r\right)\varepsilon^{-d} + \sum_{\substack{j \in S_{\eta/\varepsilon} \\ \tau_j(A) \subset \Omega_\varepsilon}} U(Z_{\tau_j(A)})\right\} \end{aligned} \quad (2.10)$$

for any $v \in L^r(\Omega)$, any $\kappa \leq \kappa_0$, $\eta > 0$, $N \in \mathbb{N}$, $Z \in \mathcal{N}_{\Omega_\varepsilon,r}(v, \kappa)$, and any $\varepsilon \leq \varepsilon_0(\kappa, \eta, N)$.

Remark 2.4. *Applying the lemma, we are only interested in the case when $r \geq p > 1$. However, it is actually valid for any $r \geq p > 0$.*

2.3. Equivalent definitions of the free energy. Before attending to the proofs of our main Theorems, we will introduce several alternative partition functions yielding the same free energy $W(L)$ as that defined in Proposition 1.2.

As suggested above, one possibility is to relax the boundary condition and to consider, instead, the configurations that are ℓ^r -close to L by taking $Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon,r}(L, \kappa))$ as defined in (1.23). The same limit is obtained also by combining both and considering the partition function $Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon,r}(L, \kappa) \cap \mathcal{N}_{\Omega_\varepsilon,R_0,\infty}(L))$.

Lemma 2.5.

Suppose that U satisfies the assumptions (A1) and (A2) with $r \geq p > 1$, $\frac{1}{r} > \frac{1}{p} - \frac{1}{d}$, and let $L : \mathbb{R}^d \rightarrow \mathbb{R}^m$ be affine and $W(L)$ be as defined in (1.13). Then:

a) *We have*

$$-\lim_{\varepsilon \rightarrow 0} \varepsilon^d |\Omega|^{-1} \log Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon,r}(L, \kappa) \cap \mathcal{N}_{\Omega_\varepsilon,R_0,\infty}(L)) = W(L). \quad (2.11)$$

In particular, the limit does not depend on κ and Ω .

b) *Using*

$$W_\kappa(L) = -\limsup_{\varepsilon \rightarrow 0} \varepsilon^d |\Omega|^{-1} \log Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon,r}(L, \kappa)), \quad (2.12)$$

we have $\lim_{\kappa \rightarrow 0} W_\kappa(L) = W(L)$.

c) *The free energy $W(L)$ satisfies the bounds $b \leq W(L) \leq B(L)$ with $b = \frac{m}{p} \log c - \log c(p, m)$, where $c(p, m) = \int_{\mathbb{R}^m} \exp(-|\xi|^p) d\xi$, and $B(L) = C[(1 + d\|L\|^r)R_0^d + 1 + C]$.*

Proof. a) Using the shorthand $\tilde{Z}_{\Lambda,\kappa}(L) = Z_{\Lambda}(\mathcal{N}_{\Lambda,r}(L, \kappa) \cap \mathcal{N}_{\Lambda, R_0, \infty}(L))$, the existence of the limit and its independence on Ω follows easily by an obvious monotonicity in κ and by standard methods with the help of approximative subadditivity (of $-\log \tilde{Z}_{\Lambda,\kappa}(L)$) similar to (1.14),

$$\log \tilde{Z}_{\Lambda,\kappa}(L) \geq \log \tilde{Z}_{\Lambda_1,\kappa}(L) + \log \tilde{Z}_{\Lambda_2,\kappa}(L) - B(L)|S(\Lambda_1, \Lambda_2)|. \quad (2.13)$$

In addition to inserting the indicator $\mathbb{1}_{\Lambda_1,L}(X)\mathbb{1}_{\Lambda_2,L}(X)$, we also observe the following inclusion, $\mathcal{N}_{\Lambda_1,r}(L, \kappa) \cap \mathcal{N}_{\Lambda_2,r}(L, \kappa) \subset \mathcal{N}_{\Lambda,r}(L, \kappa)$. Here, by $\mathcal{N}_{\Lambda_1,r}(L, \kappa) \cap \mathcal{N}_{\Lambda_2,r}(L, \kappa)$ we mean a shorthand for the set

$$\{X \in (\mathbb{R}^m)^{\Lambda} : X_{\Lambda_1} \in \mathcal{N}_{\Lambda_1,r}(L, \kappa) \text{ and } X_{\Lambda_2} \in \mathcal{N}_{\Lambda_2,r}(L, \kappa)\}. \quad (2.14)$$

Indeed, for any $X \in \mathcal{N}_{\Lambda_1,r}(L, \kappa) \cap \mathcal{N}_{\Lambda_2,r}(L, \kappa)$, we have

$$\sum_{i \in \Lambda} |X(i) - L(i)|^r \leq \kappa^r (|\Lambda_1|^{1+\frac{r}{d}} + |\Lambda_2|^{1+\frac{r}{d}}) \leq \kappa^r |\Lambda|^{1+\frac{r}{d}} \quad (2.15)$$

since, using ξ to denote $\xi = \frac{|\Lambda_1|}{|\Lambda|}$ with $1 - \xi = \frac{|\Lambda_2|}{|\Lambda|}$, we have $\xi^{1+\frac{r}{d}} + (1 - \xi)^{1+\frac{r}{d}} \leq 1$.

To prove that the limit, denoted momentarily as

$$\widetilde{W}_{\kappa}(L) = -\lim_{\varepsilon \rightarrow 0} \varepsilon^d |\Omega|^{-1} \log Z_{\Omega_{\varepsilon}}(\mathcal{N}_{\Omega_{\varepsilon},r}(L, \kappa) \cap \mathcal{N}_{\Omega_{\varepsilon}, R_0, \infty}(L)), \quad (2.16)$$

actually does not depend on κ , we first use the independence on Ω and consider the limit above with a cube Ω . Notice that the cube $\overline{\Omega}_{\varepsilon}$ obtained as the cube $\Omega_{\varepsilon/2}$ rescaled by the factor 2 consists of a disjoint union of 2^d shifts of copies of the cube Ω_{ε} , $\overline{\Omega}_{\varepsilon} = \cup_{k=1, \dots, 2^d} \tau_{i_k}(\Omega_{\varepsilon})$, $|\overline{\Omega}_{\varepsilon}| = 2^d |\Omega_{\varepsilon}|$. We have $\cap_{k=1, \dots, 2^d} \mathcal{N}_{\tau_{i_k}(\Omega_{\varepsilon}),r}(L, \kappa) \subset \mathcal{N}_{\overline{\Omega}_{\varepsilon},r}(L, \kappa/2)$. Indeed, for any $X \in \cap_{k=1, \dots, 2^d} \mathcal{N}_{\tau_{i_k}(\Omega_{\varepsilon}),r}(L, \kappa)$, similarly as in (2.15), we have

$$\sum_{i \in \overline{\Omega}_{\varepsilon}} |X(i) - L(i)|^r \leq 2^d \kappa^r |\Omega_{\varepsilon}|^{1+\frac{r}{d}} = 2^d \kappa^r (2^{-d} |\overline{\Omega}_{\varepsilon}|)^{1+\frac{r}{d}} = (\frac{\kappa}{2})^r |\overline{\Omega}_{\varepsilon}|^{1+\frac{r}{d}}. \quad (2.17)$$

As a result,

$$\log \tilde{Z}_{\overline{\Omega}_{\varepsilon}, \kappa/2}(L) = \log \tilde{Z}_{\Omega_{\varepsilon/2}, \kappa/2}(L) \geq 2^d \log \tilde{Z}_{\Omega_{\varepsilon}, \kappa} - B(L) 2^d |\partial \Omega| \varepsilon^{-d+1} R_0. \quad (2.18)$$

Multiplying by $-(\varepsilon/2)^d |\Omega|^{-1}$ and taking the limit $\varepsilon \rightarrow 0$, we get $\widetilde{W}_{\kappa/2}(L) \leq \widetilde{W}_{\kappa}(L)$. On the other hand, $\widetilde{W}_{\kappa/2}(L) \geq \widetilde{W}_{\kappa}(L)$ since $\widetilde{W}_{\kappa}(L)$ is clearly decreasing in κ .

Combining discrete Poincaré inequality with the assumption (A1), we see that for any fixed K and κ , we have $\mathcal{N}_{\Omega_{\varepsilon},r}(L, \kappa)^c \subset \mathcal{M}(K)$ for sufficiently small ε . Then, by exponential tightness, for any fixed δ and sufficiently small ε ,

$$Z_{\Omega_{\varepsilon}}(\mathcal{N}_{\Omega_{\varepsilon},r}(L, \kappa) \cap \mathcal{N}_{\Omega_{\varepsilon}, R_0, \infty}(L)) \geq (1 - \delta) Z_{\Omega_{\varepsilon}}(\mathcal{N}_{\Omega_{\varepsilon}, R_0, \infty}(L)), \quad (2.19)$$

implying that the limiting value $\widetilde{W}_\kappa = \widetilde{W}$ satisfies, for any δ , the inequalities

$$\begin{aligned} W(L) \leq \widetilde{W}(L) &= -\lim_{\varepsilon \rightarrow 0} \varepsilon^d |\Omega|^{-1} \log Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon, r}(L, \kappa) \cap \mathcal{N}_{\Omega_\varepsilon, R_0, \infty}(L)) \leq \\ &\leq -\lim_{\varepsilon \rightarrow 0} \varepsilon^d |\Omega|^{-1} \log[(1 - \delta) Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon, R_0, \infty}(L))] = W(L). \end{aligned} \quad (2.20)$$

b) Choosing $v = L$ and $Z = L$ ($Z(i) = L(i)$ for each $i \in \Omega_\varepsilon$) in the interpolation lemma, we get

$$\widetilde{W}(L) \geq W_\kappa(L) \geq \widetilde{W}(L) - \mathcal{C} \left(\frac{b + W_\kappa(L)}{N} + \eta + \left(\frac{N\kappa}{\eta} \right)^r + \eta \|L\|^r C_\partial |\partial\Omega| \right) \quad (2.21)$$

yielding

$$\widetilde{W}(L) \geq \lim_{\kappa \rightarrow 0} W_\kappa(L) \geq \widetilde{W}(L) - \mathcal{C} \left(\frac{b + \lim_{\kappa \rightarrow 0} W_\kappa(L)}{N} + \eta(1 + \|L\|^r C_\partial |\partial\Omega|) \right) \quad (2.22)$$

for arbitrarily small η and arbitrarily large N .

c) The lower bound follows from the inequality

$$\widetilde{Z}_{\Omega_\varepsilon, \kappa}(L) \leq \omega(m) \left(c^{-m/p} c(p, m) \right)^{|\Omega_\varepsilon|}, \quad (2.23)$$

obtained with help of (A1) and the bound from technical Lemma A.1 a) proven in Appendix. For the upper bound, we just take into account that $\mathcal{N}_{\Omega_\varepsilon, \infty}(Z) \subset \mathcal{N}_{\Omega_\varepsilon, r}(v, \kappa)$ for sufficiently small ε (cf. Remark 2.2), to get

$$\widetilde{Z}_{\Omega_\varepsilon, \kappa}(L) \geq \omega(m)^{|\Omega_\varepsilon|} \exp(-C(C(1 + dR_0^d \|L\|^r) + (1 + R_0^d) |\Omega_\varepsilon|)) \quad (2.24)$$

similarly as in (2.6) with the bound $U(L) \leq C(1 + dR_0^d \|L\|^r)$ resulting from (A2) (in the form from Remark 1.1). \square

Finally, we can enforce a version of approximate periodic boundary conditions yielding again the same free energy $W(L)$. Namely, consider the sets

$$\mathcal{N}^{\text{per}, \varepsilon}(L) = \{X : \mathbb{Z}^d \rightarrow \mathbb{R}^m : |X(i + \frac{\mathbf{e}_j}{\varepsilon}) - X(i) - \frac{L(\mathbf{e}_j)}{\varepsilon}| \leq 2 \ \forall i \in \mathbb{Z}^d, j = 1, \dots, d\}, \quad (2.25)$$

and

$$\mathcal{N}_r^{\text{per}, \varepsilon}(L, \kappa) = \{X \in \mathcal{N}^{\text{per}, \varepsilon}(L) : \|\Pi_\varepsilon(X) - L\|_{L^r([0, 1]^d)} \leq \kappa\}, \quad (2.26)$$

and define

$$Z_{[0, 1]^d_{\varepsilon, R_0}}(\mathcal{N}^{\text{per}, \varepsilon}(L)) = \int_{\mathcal{N}^{\text{per}, \varepsilon}(L)} \exp\{-H_{[0, 1]^d_{\varepsilon, R_0}}(X)\} \prod_{i \in [0, 1]^d_{\varepsilon, R_0}} dX(i) \quad (2.27)$$

and, similarly, also $Z_{[0, 1]^d_{\varepsilon, R_0}}(\mathcal{N}_r^{\text{per}, \varepsilon}(L, \kappa))$. Here, we use $[0, 1]^d_{\varepsilon, R_0}$ to denote the set

$$\{i \in \mathbb{Z}^d : i_j \in [-R_0, \varepsilon^{-1} + R_0], j = 1, \dots, d\}. \quad (2.28)$$

Observing that

$$\begin{aligned} Z_{[0,1]_{\varepsilon,R_0}^d}(\mathcal{N}_{[0,1]_{\varepsilon,R_0}^d,r}(L, \kappa) \cap \mathcal{N}_{[0,1]_{\varepsilon,R_0}^d,R_0,\infty}(L)) &\leq Z_{[0,1]_{\varepsilon,R_0}^d}(\mathcal{N}_r^{\text{per},\varepsilon}(L, \kappa)) \leq \\ &\leq Z_{[0,1]_{\varepsilon,R_0}^d}(\mathcal{N}_{[0,1]_{\varepsilon,R_0}^d,r}(L, \kappa)) \end{aligned} \quad (2.29)$$

and applying the preceding lemma, we get

$$W(L) = -\lim_{\kappa \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \varepsilon^d \log Z_{[0,1]_{\varepsilon,R_0}^d}^{\text{per}}(\mathcal{N}_r^{\text{per},\varepsilon}(L, \kappa)). \quad (2.30)$$

Similarly as in Lemma 2.5 (b), we obtain the same limit also with $Z_{[0,1]_{\varepsilon,R_0}^d}(\mathcal{N}^{\text{per},\varepsilon}(L))$:

Lemma 2.6.

Suppose that (A1) and (A2) hold with $r \geq p > 1$ and $\frac{1}{r} > \frac{1}{p} - \frac{1}{d}$. Then the free energy $W(L)$ from Proposition 1.2 equals

$$W(L) = -\lim_{\kappa \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \varepsilon^d \log Z_{[0,1]_{\varepsilon,R_0}^d}^{\text{per}}(\mathcal{N}_r^{\text{per},\varepsilon}(L, \kappa)) = -\lim_{\varepsilon \rightarrow 0} \varepsilon^d \log Z_{[0,1]_{\varepsilon,R_0}^d}^{\text{per}}(\mathcal{N}^{\text{per},\varepsilon}(L)). \quad (2.31)$$

2.4. Proof of Large Deviation Principle. To prove Theorem 1.6, we begin by considering, for any $\varrho > 0$ and $z \in Q(\varrho) = [-\frac{\varrho}{2}, \frac{\varrho}{2}]^d$, the lattice

$$\mathcal{L}_{\varrho,z} = (\varrho\mathbb{Z})^d + z = \{x \in (\varrho\mathbb{Z})^d + z\}. \quad (2.32)$$

Our strategy will be to approximate the integrals $Z_{\Omega_\varepsilon}(O(v))$ over suitably chosen neighbourhoods $O(v)$, $v \in W^{1,p}(\Omega)$, by a product of contributions over cubes obtained from $Q(\varrho)$ by shifts from $\mathcal{L}_{\varrho,z}$. Here ϱ and z will be chosen so that the function v is, on each cube $x + Q(\varrho)$ for which $x + Q(\varrho) \subset \Omega$, well approximated by its linear part $\mathcal{L}_x v$ defined at x by $\mathcal{L}_x v(y) = \nabla v(x) \cdot y + \int_{x+Q(\varrho)} v(t) dt$ and, in the same time, the sum of the contributions $W(\nabla v(x))$ over the linear patches is well represented by the integral $\int_\Omega W(\nabla v(x)) dx$.

To show that such a choice (of ϱ and z) is possible, we will use the following “blow up” lemma (the Corollary below) with a function $f(x)$ related to an approximation of $W(\nabla v(x))$ and the functions $v_{x,\varrho}$ representing the difference $v - \mathcal{L}_x v$; explicitly, we define

$$v_{x,\varrho}(y) = \frac{1}{\varrho}(v(x + \varrho y) - \mathcal{L}_x v(\varrho y)) = \frac{v(x + \varrho y) - \int_{x+Q(\varrho)} v(t) dt}{\varrho} - \nabla v(x) \cdot y \quad (2.33)$$

for any $x \in \mathcal{L}_{\varrho,z}$ and any $y \in Q = Q(1)$. For $v \in W^{1,p}(\Omega)$, the function $v_{x,\varrho}(\cdot)$ is considered as belonging to $L^p(\Omega, W^{1,p}(\Omega))$.

Lemma 2.7. *Let $r \geq p > 1$, $\frac{1}{r} > \frac{1}{p} - \frac{1}{d}$, and let $v \in W_0^{1,p}(\mathbb{R}^d)$. Then there exists a function $\omega_v : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\lim_{\varrho \rightarrow 0} \omega_v(\varrho) = 0$ and*

- a) $\int_{Q(\varrho)} \sum_{x \in \mathcal{L}_{\varrho,z}} \varrho^d (\int_{Q(1)} |\nabla v_{x,\varrho}(y)|^p dy) dz \leq \omega_v(\varrho),$
- b) $\int_{Q(\varrho)} \sum_{x \in \mathcal{L}_{\varrho,z}} \varrho^d (\int_{Q(1)} |v_{x,\varrho}(y)|^r dy)^{p/r} dz \leq \omega_v(\varrho).$

Proof. a) Notice first that for any $\omega > 0$ we can choose ϱ sufficiently small, to get

$$\int_{\mathbb{R}^d} \int_Q |\nabla v_{x,\varrho}(y)|^p dy dx = \int_{\mathbb{R}^d} \int_Q |\nabla v(x + \varrho y) - \nabla v(x)|^p dy dx < \omega \quad (2.34)$$

by Lebesgue differentiation theorem. Rewriting the integral $\int_{\mathbb{R}^d} \int_Q |\nabla v_{x,\varrho}(y)|^p dy dx$ in the form of the sum $\int_{Q(\varrho)} \sum_{x \in \mathcal{L}_{\varrho,z}} \int_Q |\nabla v_{x,\varrho}(y)|^p dy dz$, we get

$$\frac{1}{\varrho^d} \int_{Q(\varrho)} \left(\varrho^d \sum_{x \in \mathcal{L}_{\varrho,z}} \int_Q |\nabla v_{x,\varrho}(y)|^p dy \right) dz < \omega. \quad (2.35)$$

b) Follows from a) by Sobolev imbedding. \square

Corollary 2.8. *Let $v \in W_0^{1,p}(\mathbb{R}^d)$, $\delta > 0$, $f \in L_1(\mathbb{R}^d)$, and let $\ell < \int_{\mathbb{R}^d} f(x) dx$. Then there exists a constant $\varrho_0 = \varrho_0(v, f, \ell, \delta)$ and for each $\rho \leq \rho_0$ a point $z \in Q(\varrho)$ such that*

$$\sum_{x \in \mathcal{L}_{\varrho,z}} \varrho^d \int_Q |\nabla v_{x,\varrho}(y)|^p dy < \delta \quad \text{and} \quad \varrho^d \sum_{x \in \mathcal{L}_{\varrho,z}} f(x) > \ell. \quad (2.36)$$

Proof. Interpreting the integral in (2.35) as the mean over $Q(\varrho)$ of the function in the brackets and using $\mathcal{M}_\delta \subset Q(\varrho)$ to denote the set of points for which the first inequality in (2.36) is not valid,

$$\mathcal{M}_\delta = \{z \in Q(\varrho) \mid \varrho^d \sum_{x \in \mathcal{L}_{\varrho,z}(\Omega)} \int_Q |\nabla v_{x,\varrho}(y)|^p dy \geq \delta\}, \quad (2.37)$$

we can apply Markov's inequality to get

$$|\mathcal{M}_\delta| \leq \frac{\omega_v(\rho) \varrho^d}{\delta}. \quad (2.38)$$

On the other hand, assuming without loss of generality that $f \leq K \mathbb{1}_{Q(R)}$ for some (large) K and R and denoting $F(z) = \varrho^d \sum_{x \in \mathcal{L}_{\varrho,z}} f(x)$, we have $F \leq KR^d$ with the mean over $Q(\varrho)$ satisfying

$$\int F(z) dz = \int_{Q(\varrho)} \varrho^{-d} F(z) dz = \int f(y) dy > \ell. \quad (2.39)$$

Denoting

$$\overline{\mathcal{M}}_\ell = \{z \in Q(\varrho) \mid F(z) \leq \ell\}, \quad (2.40)$$

we get

$$\int f(y) dy = \int F(z) dz \leq \ell \varrho^{-d} |\overline{\mathcal{M}}_\ell| + KR^d (1 - \varrho^{-d} |\overline{\mathcal{M}}_\ell|). \quad (2.41)$$

Hence,

$$1 - \varrho^{-d} |\overline{\mathcal{M}}_\ell| \geq \frac{\int f(y) dy - \ell}{KR^d - \ell}. \quad (2.42)$$

A point z satisfying simultaneously both bounds in (2.36) thus exists once $1 - \varrho^{-d}|\overline{\mathcal{M}}_\ell| - \frac{\omega_v(\varrho)}{\delta} > \epsilon$ for a fixed ϵ and ϱ small. For this to hold, it is enough to choose ω_v (and corresponding ϱ) sufficiently small. \square

Theorem 1.6 a) follows directly from the following lemma.

Lemma 2.9. *For every $\delta, \kappa, M \in (0, \infty)$ and any $v \in W^{1,p}(\Omega)$ with $r \geq p > 1$, $\frac{1}{r} > \frac{1}{p} - \frac{1}{d}$, there exists $\tilde{\kappa}$ such*

$$Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon, r}(v, \tilde{\kappa})) \leq \exp\left\{\varepsilon^{-d}\left(-\int_{\Omega} (W_\kappa(\nabla v(x)) \wedge M) dx + \delta\right)\right\} \quad (2.43)$$

for sufficiently small ε .

Remark 2.10. *Whenever $\int_{\Omega} W(\nabla v(x)) dx < \infty$, we infer by Lebesgue theorem that*

$$Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon, r}(v, \tilde{\kappa})) \leq \exp\left\{\varepsilon^{-d}\left(-\int_{\Omega} W(\nabla v(x)) dx + \delta\right)\right\}. \quad (2.44)$$

If $\int_{\Omega} W(\nabla v(x)) dx = \infty$, we can show that for any M there exists $\varepsilon(M)$ so that

$$Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon, r}(v, \tilde{\kappa})) \leq \exp\{-\varepsilon^{-d}M\} \quad (2.45)$$

for $\varepsilon < \varepsilon(M)$.

Proof. Replacing v by an extension to $W^{1,p}(\mathbb{R}^d)$ with compact support, we apply Lemma 2.8 with $f(x) = (W_\kappa(\nabla v(x)) \wedge M) \mathbb{1}_\Omega(x)$. Thus, for any constant $\tilde{\delta} > 0$ and any ϱ_0 , there exists $\varrho < \varrho_0$ and a point $z \in Q(\varrho)$ such that

$$\sum_{x \in \mathcal{L}_{\varrho, z}} \varrho^d \int_Q |\nabla v_{x, \varrho}(y)|^p dy < \tilde{\delta} \quad \text{and} \quad (2.46)$$

$$\varrho^d \sum_{x \in \mathcal{L}_{\varrho, z}} (W_\kappa(\nabla v(x)) \wedge M) \mathbb{1}_\Omega(x) > \int_{\Omega} (W_\kappa(\nabla v(z)) \wedge M) dz - \tilde{\delta}. \quad (2.47)$$

Now, let us consider the vector $\kappa = \{\kappa_x, x \in \mathcal{L}_{\varrho, z}\}$ with $\kappa_x^p = \int_Q |\nabla v_{x, \varrho}(y)|^p dy$, and the neighbourhood

$$O_\kappa(v) = \bigcap_{x \in \mathcal{L}_{\varrho, z}, \tau_x(Q(\varrho)) \cap \Omega \neq \emptyset} \mathcal{N}_{\tau_x(Q(\varrho)) \cap \Omega_\varepsilon, r}(\mathcal{L}_x v, \kappa_x). \quad (2.48)$$

Cf. (1.28) for the definition of $\mathcal{N}_{\Lambda, r}(v, \kappa)$.

Using (A1), we have $H_{\Omega_\varepsilon}(X) \geq \sum_{x \in \mathcal{L}_{\varrho, z}(\Omega)} H_{\tau_x(Q(\varrho))_\varepsilon \cap \Omega_\varepsilon}(X)$. Thus

$$Z_{\Omega_\varepsilon}(O_\kappa(v)) \leq \prod_{x \in \mathcal{L}_{\varrho, z}} Z_{\tau_x(Q(\varrho))_\varepsilon}(\mathcal{N}_{\tau_x(Q(\varrho))_\varepsilon \cap \Omega_\varepsilon, r}(\mathcal{L}_x v, \kappa_x)). \quad (2.49)$$

Above, we take $Z_{\tau_x(Q(\varrho))_\varepsilon}(\mathcal{N}_{\tau_x(Q(\varrho))_\varepsilon \cap \Omega_\varepsilon, r}(\mathcal{L}_x v, \kappa_x)) = 1$ whenever $\tau_x(Q(\varrho))_\varepsilon \cap \Omega_\varepsilon = \emptyset$.

Taking now \limsup of the appropriately rescaled logarithm of (2.49), we get

$$\begin{aligned}
 \limsup_{\epsilon \rightarrow 0} \epsilon^d \log Z_{\Omega_\epsilon}(O_\kappa(v)) &\leq \sum_{x \in \mathcal{L}_{\varrho,z}} \limsup_{\epsilon \rightarrow 0} \epsilon^d \log Z_{\tau_x(Q(\varrho))_\epsilon}(\mathcal{N}_{\tau_x(Q(\varrho))_\epsilon \cap \Omega_\epsilon, r}(\mathcal{L}_x v, \kappa_x)) = \\
 &= -\varrho^d \sum_{x \in \mathcal{L}_{\varrho,z}} W_{\kappa_x}(\nabla v(x)) \mathbb{1}_\Omega(x) \leq -\varrho^d \sum_{x \in \mathcal{L}_{\varrho,z}} (W_{\kappa_x}(\nabla v(x)) \wedge M) \mathbb{1}_\Omega(x) \leq \\
 &\leq -\varrho^d \sum_{x \in \mathcal{L}_{\varrho,z}} (W_\kappa(\nabla v(x)) \wedge M) \mathbb{1}_\Omega(x) - \varrho^d \sum_{\substack{x \in \mathcal{L}_{\varrho,z} \\ \kappa_x > \kappa}} (W_{\kappa_x}(\nabla v(x)) \wedge M - W_\kappa(\nabla v(x)) \wedge M) \mathbb{1}_\Omega(x).
 \end{aligned} \tag{2.50}$$

The absolute value of each term in the last last sum can be bounded by $M + |b|$ with b the lower bound from Lemma 2.5. In the same time, the number of terms n_κ for which $\kappa_x > \kappa$ is, in view of the bound $\sum \varrho^d \kappa_x^p < \tilde{\delta}$, bounded by $n_\kappa \leq \varrho^{-d} \frac{\tilde{\delta}}{\kappa^p}$.

In summary, observing that for sufficiently small $\tilde{\kappa}$ the set $\mathcal{N}_{\Lambda, r}(v, \tilde{\kappa})$ is contained in the intersection $O_\kappa(v)$ of a finite number of open sets, we are getting, for sufficiently small ϵ ,

$$\epsilon^d \log Z_{\Omega_\epsilon}(O_\kappa(v)) \leq - \int_\Omega W_\kappa(\nabla v(x)) dx + (M + |b|) \frac{\tilde{\delta}}{\kappa^p} + \tilde{\delta} \tag{2.51}$$

obtaining the claim by choosing sufficiently small ρ and $\tilde{\delta}$. \square

For the lower bound, Theorem 1.6 b), we have to use Interpolation Lemma again (more precisely, we use Lemma 2.5(a) that is based on it).

Lemma 2.11. *a) For every $\delta, \kappa \in (0, \infty)$ and any $v \in W^{1,p}(\Omega)$, we have*

$$Z_{\Omega_\epsilon}(\mathcal{N}_{\Lambda, r}(v, \kappa)) \geq \exp\left\{\epsilon^{-d} \left(- \int_\Omega W_{\kappa/2}(\nabla w(x)) dx - \delta\right)\right\} \tag{2.52}$$

if w is a piecewise linear function such that $\|w - v\|_r \leq \frac{\kappa}{2} |\Omega|^{\frac{1}{r} + \frac{1}{d}}$ and ϵ is sufficiently small.

b) For every $\delta, \kappa \in (0, \infty)$ and any $v \in W^{1,r}(\Omega)$, we have

$$Z_{\Omega_\epsilon}(\mathcal{N}_{\Lambda, r}(v, \kappa)) \geq \exp\left\{\epsilon^{-d} \left(- \int_\Omega W(\nabla v(x)) dx - \delta\right)\right\} \tag{2.53}$$

for sufficiently small ϵ .

Proof. a) For the first claim we first observe that

$$\bigcap_j (\mathcal{N}_{T_j, r}(w, \kappa/2) \cap \mathcal{N}_{T_j, R_0, \infty}(w)) \subset \mathcal{N}_{\Omega, r}(v, \kappa) \tag{2.54}$$

with $\{T_j\}$ denoting a triangulation consistent with piecewise linearity of w . Using the bound $U(X_{\tau_i(A)}) \leq C(1 + U(X_{w, \epsilon}) + R_0^d)$ whenever $\tau_i(A)$ is reaching over the

boundaries of the linear parts of w and then applying Lemma 2.5(a) to evaluate each term $Z_{T_j}(\mathcal{N}_{T_j,r}(w, \kappa/2) \cap \mathcal{N}_{T_j,R_0,\infty}(w))$, we get the sought bound with a constant proportional to ε^{-d+1} which is smaller than δ for sufficiently small ε .

b) For the second claim, we first notice (see Lemma 2.5(c)) that $W(L) \leq C(1 + \|L\|^r)$. It follows that, if w_n is a sequence of piecewise linear functions such that $\int |\nabla(w_n(x) - v(x))|^r dx \rightarrow 0$, then $W_{\kappa/2}(\nabla w_n(x))$ is equiintegrable. Using the bound $W_{\kappa/2}(L) \leq W_{\kappa/4}(\tilde{L})$ valid for $\|L - \tilde{L}\| < \kappa/4$, we conclude that

$$\lim \int W_{\kappa/2}(\nabla w_n(x)) dx \leq \int W_{\kappa/4}(\nabla v(x)) dx \leq \int W(\nabla v(x)) dx. \quad (2.55)$$

□

2.5. Proof of Proposition 1.4. Consider $v \in W_0^{1,r}(\Omega) + L$. According to Theorem 1.6 b), we have $\lim_{\kappa \rightarrow 0} \liminf_{\varepsilon \rightarrow 0} F_{\kappa,\varepsilon}(v) \leq \frac{1}{|\Omega|} \int_{\Omega} W(\nabla v(x)) dx$ with $F_{\kappa,\varepsilon}(v) = -\varepsilon^d |\Omega|^{-1} \log Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon,r}(v, \kappa))$. Using the obvious inequality

$$Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon,r}(v, 2\kappa) \cap \mathcal{N}_{\Omega_\varepsilon,R_0,\infty}(L)) \leq Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon,R_0,\infty}(L)) \quad (2.56)$$

and Interpolation Lemma, we get

$$Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon,r}(v, \kappa)) \leq Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon,R_0,\infty}(L)) \exp \left\{ \varepsilon^{-d} \mathcal{C} \left(\frac{b + F_{\kappa,\varepsilon}(v)}{N} + \eta + \left(\frac{N\kappa}{\eta} \right)^r + \eta \|L\|^r C_\partial |\partial \Omega| \right) \right\}. \quad (2.57)$$

In view of Proposition 1.2 thus

$$\lim_{\kappa \rightarrow 0} \liminf_{\varepsilon \rightarrow 0} F_{\kappa,\varepsilon}(v) = W(L), \quad (2.58)$$

implying the claim

$$W(L) \leq \frac{1}{|\Omega|} \int_{\Omega} W(\nabla v(x)) dx. \quad (2.59)$$

□

2.6. Non-convexity of the free energy. Let us briefly discuss the fact that the free energy $W(L)$ may be, in general, a non-convex function of L (Remark 1.5). The idea hinges on the fact that an addition, to the original Hamiltonian $H^{(0)}$, of a term in the form of a hugely non-convex discrete null Lagrangian leads to a directly computable addition to the original free energy W_0 yielding a non-convex sum $W(L)$. It suffices to assume that the free energy W_0 is bounded from above and below, $b \leq W_0(L) \leq B$, for all L with $\|L\| \leq 1$. An example might be the potential $U(X) = |\nabla X(0)|^p$ for which $W_0(L) \sim \|L\|^p$.

In more details, consider, for simplicity, the case $d = m = 2$. Let Q be a unit square $Q = (i_0, i_1, i_2, i_3)$ in \mathbb{Z}^2 (with $i_0 = (0, 0)$, $i_1 = (1, 0)$, $i_2 = (0, 1)$, and $i_3 = (1, 1)$) and, for any $X \in (\mathbb{R}^m)^Q$, let $V(X_Q)$ be defined by

$$V(X_Q) = \frac{1}{2} \det(X(i_1) - X(i_0), X(i_2) - X(i_0)) + \frac{1}{2} \det(X(i_1) - X(i_3), X(i_2) - X(i_3)). \quad (2.60)$$

Geometrically, $V(X_Q)$ yields the area of the rectangle $(X(i_0), X(i_1), X(i_3), X(i_2))$. In particular, for an affine map L , $V(L)$ is the area of the deformed square $L(Q)$. Thus, $V(\text{id}) = 1$ for the identity map id , $\text{id}(i) = i$, and $V(L^{(0)}) = 0$ for the zero map $L^{(0)}$, $L^{(0)}(i) = 0$.

Consider the Hamiltonian

$$H_{\Omega_\varepsilon}(X) = H_{\Omega_\varepsilon}^{(0)}(X) + M \sum_{j \in \mathbb{Z}^d: \tau_j(Q) \subset \Omega_\varepsilon} (1 - V(X_{\tau_j(Q)})) = H_{\Omega_\varepsilon}^{(0)}(X) + \begin{cases} 0 & \text{for } L = \text{id}, \\ M & \text{for } L = L^{(0)}, \end{cases} \quad (2.61)$$

where $H_{\Omega_\varepsilon}^{(0)}(X)$ is the original Hamiltonian and $M > 0$ is a constant. The crucial point is that the term V is a discrete null Lagrangian (see e.g. [7]): the value of the additional term

$$H_{\Omega_\varepsilon}^*(X) = M \sum_{j \in \mathbb{Z}^d: \tau_j(Q) \subset \Omega_\varepsilon} (1 - V(X_{\tau_j(Q)})) \quad (2.62)$$

depends only on X in the boundary layer, $H_{\Omega_\varepsilon}^*(X) = H_{\Omega_\varepsilon}^*(\bar{X})$ if $X|_{\partial_{R_0}\Omega_\varepsilon} = \bar{X}|_{\partial_{R_0}\Omega_\varepsilon}$. More precisely, $H_{\Omega_\varepsilon}^*(X)$ equals $M(\text{vol}(\text{id}) - \text{vol}(X))$, where $\text{vol}(X)$ is the signed volume of the envelope of the set points $X(i), i \in \Omega_\varepsilon$.

We have

Lemma 2.12. *Let U be a potential whose corresponding free energy W_0 is bounded from above and below, $W_0(L) \in (b, B)$, for every L such that $\|L\| \leq 1$. Then the free energy W corresponding to the Hamiltonian $H^{(0)} + H^*$ is non-convex for M sufficiently large.*

Proof. Consider $L_1 = \text{id}$ and $L_2 = -\text{id}$. For any $X \in \mathcal{N}_{\Omega_\varepsilon, R_0, \infty}(L_1)$, we have $H_{\Omega_\varepsilon}^*(X) = H_{\Omega_\varepsilon}^*(L_1) + O(\varepsilon^{d-1})\|L_1\| = O(\varepsilon^{d-1})$ since the volume spanned by $X(i), i \in \Omega_\varepsilon$ differs from the volume spanned by Ω_ε at most by $O(\varepsilon^{d-1})\|L_1\|$. Similarly $H_{\Omega_\varepsilon}^*(X) = O(\varepsilon^{d-1})$ for any $X \in \mathcal{N}_{\Omega_\varepsilon, R_0, \infty}(L_2)$. Thus $W(L_1) = W^{(0)}(L_1)$ and $W(L_2) = W^{(0)}(L_2)$. Given that $\frac{1}{2}L_1 + \frac{1}{2}L_2 = L^{(0)}$ and $H_{\Omega_\varepsilon}^*(X) = M|\Omega_\varepsilon| + O(\varepsilon)$ for every $X \in \mathcal{N}_{\Omega_\varepsilon, R_0, \infty}(L_0)$, we get $W(L^{(0)}) = W^{(0)}(L^{(0)}) + M \geq b + M > B \geq \frac{1}{2}W^{(0)}(L_1) + \frac{1}{2}W^{(0)}(L_2) = \frac{1}{2}W(L_1) + \frac{1}{2}W(L_2)$ once $M > B - b$. \square

2.7. Proof of Theorem 1.10. We will use a particular case of the following Lemma formulated in an abstract setting. It is based on the following two standard facts.

(1) Let \mathcal{X} be a topological space, $\mathcal{K}_\ell \subset \subset \mathcal{X}$ a sequence of its compact separable subspaces, and $\varepsilon_\ell \rightarrow 0$ a sequence of positive numbers. Then the set of Borel probability measures with uniform tightness condition,

$$\mathcal{M}_{(\varepsilon_\ell)} = \{\alpha \in BC(\mathcal{X})^*: 0 \leq \alpha, \alpha(1) = 1, \alpha(\mathcal{X} \setminus \mathcal{K}_\ell) \leq \varepsilon_\ell\} \quad (2.63)$$

is weakly compact. Here, as usually, $\alpha(\mathcal{X} \setminus \mathcal{K}_\ell) = \sup\{\alpha(\varphi) : \varphi \in C(\mathcal{X}), \varphi \leq \mathbb{1}_{\mathcal{X} \setminus \mathcal{K}_\ell}\}$.

Moreover, if we have a sequence μ_n of Borel probability measures on \mathcal{X} such that $\mu_n(\mathcal{X} \setminus \mathcal{K}_\ell) \leq \varepsilon_\ell$ for all $n > n(\ell)$, then a subsequence converges weakly to a Borel probability measure $\mu \in \mathcal{M}_{(\varepsilon_\ell)}$.

(2) Let \mathcal{X}_1 and \mathcal{X}_2 be topological spaces and $\mathcal{K}_{1,\ell} \subset \subset \mathcal{X}_1$ and $\mathcal{K}_{2,\ell} \subset \subset \mathcal{X}_2$ be sequences of separable compact subspaces and $\mu \in BC(\mathcal{X}_1 \times \mathcal{X}_2)^*$ be such that $\lim_{\ell \rightarrow \infty} \mu(\mathcal{X}_1 \times \mathcal{X}_2 \setminus \mathcal{K}_{1,\ell} \times \mathcal{K}_{2,\ell}) = 0$. Then there exists a weakly measurable map $\nu : \mathcal{X}_1 \rightarrow BC(\mathcal{X}_2)^*$ so that

$$\mu(\varphi) = \int_{\mathcal{X}_1} \nu_x(\varphi(x, \cdot)) d\mu_1, \quad (2.64)$$

where μ_1 is the marginal of μ . Moreover, $\nu_x(1) = \lim \nu_x(\mathcal{K}_{2,\ell})$ for almost all x .

Now, let (S_n, μ_n, Σ_n) be a sequence of probabilities, $\mathcal{X}_1, \mathcal{X}_2$, and \mathcal{X}_3 be topological spaces and $\mathcal{K}_{k,\ell} \subset \subset \mathcal{X}_k$, $k = 1, 2, 3$, be sequences of separable compact subspaces, and let λ be a Borel probability measure on \mathcal{X}_3 . Further, let a sequence of mappings $\mathcal{T}_n : S_n \times \mathcal{X}_3 \rightarrow \mathcal{X}_1 \times \mathcal{X}_2$ be given that are measurable with respect to the Borel σ -algebras on $\mathcal{X}_1, \mathcal{X}_2$.

We say that a sequence \mathcal{T}_n fulfills a *uniform tightness condition* (with respect to probabilities $\mu_n \times \lambda$ and a sequence (ε_ℓ) , $\varepsilon_\ell \rightarrow 0$) if

$$\lambda(\mathcal{X}_3 \setminus \mathcal{K}_{3,\ell}) + (\mu_n \times \lambda)(\mathcal{T}_n^{-1}((\mathcal{X}_1 \times \mathcal{X}_2) \setminus (\mathcal{K}_{1,\ell} \times \mathcal{K}_{2,\ell}))) \leq \varepsilon_\ell \quad (2.65)$$

for every ℓ and $n \geq n(\ell)$.

In this setting, the observations (1) and (2) lead to the following claim.

Lemma 2.13. *Given a sequence \mathcal{T}_n fulfilling a uniform tightness condition, there exists a subsequence $n_k \rightarrow \infty$, Borel probability measures γ on \mathcal{X}_1 and λ on \mathcal{X}_3 such that*

$$\gamma(\mathcal{X}_1 \setminus \mathcal{K}_{1,\ell}) \leq \varepsilon_\ell \text{ and } \lambda(\mathcal{X}_3 \setminus \mathcal{K}_{3,\ell}) \leq \varepsilon_\ell \text{ for all } \ell, \quad (2.66)$$

and a mapping $\nu : \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow \mathcal{P}(\mathcal{X}_2)$ that is weakly measurable (with respect to the weak topology of $(BC(\mathcal{X}_2))^$) satisfying*

$$0 \leq \nu_{x_1, x_3}, \nu_{x_1, x_3}(1) = 1 \quad \text{and} \quad \int_{\mathcal{X}_1 \times \mathcal{X}_3} \nu_{x_1, x_3}(\mathcal{X}_2 \setminus \mathcal{K}_{2,\ell}) d\gamma(x_1) d\lambda(x_3) \leq \varepsilon_\ell \quad (2.67)$$

for almost all x_1 and x_3 , such that

$$\begin{aligned} \lim_{k \rightarrow \infty} \int \varphi(\mathcal{T}_{n_k}(\omega, x_3), x_3) d\mu_{n_k}(\omega) d\lambda(x_3) = \\ = \int_{\mathcal{X}_1 \times \mathcal{X}_3} \left[\int_{\mathcal{X}_2} \varphi(x_1, x_2, x_3) d\nu_{x_1, x_3}(x_2) \right] d\gamma(x_1) d\lambda(x_3). \end{aligned} \quad (2.68)$$

for any bounded and continuous test function φ on $\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$.

We will apply the above Lemma in the following situation.

We take $\mathcal{X}_1 = W_0^{1,p}(\Omega)$, $\mathcal{X}_2 = BC(S)^*$, and $\mathcal{X}_3 = \Omega$ (recall that $S = (\mathbb{R}^m)^{\mathbb{Z}^d}/\mathbb{R}^m$). Further, we consider the sets

$$\mathcal{K}_{1,\ell} = \{v \in W_0^{1,p}(\Omega) + u : \|\nabla v\|_p \leq \ell\}. \quad (2.69)$$

Note that by the Poincaré inequality, $\mathcal{K}_{1,\ell}$ is bounded in the norm topology of $W_{1,p}(\Omega)$. To define $\mathcal{K}_{2,\ell}$, we first introduce the sets

$$B_{\Lambda,\ell} = \{X \in S : \sum_{i \in \Lambda} |\nabla X(i)|^p \leq \ell\} \quad (2.70)$$

and

$$\tilde{\mathcal{K}}_{2,\ell} = \{\mu \in BC(S)^* : 0 \leq \mu, |\mu| \leq 1, \text{supp } \mu \subset B_{\Lambda_N, 2^N |\Lambda_N| \ell} \text{ for every } N\}. \quad (2.71)$$

Here, (Λ_N) is the sequence of sets $\Lambda_N = [-N, N]^d \cap \mathbb{Z}^d$. Then

$$\begin{aligned} \mathcal{K}_{2,\ell} &= \{\mu \in BC(S)^* : 0 \leq \mu, |\mu| \leq 1, \text{ for any } k \\ &\quad \text{there exists } \mu_k \in \tilde{\mathcal{K}}_{2, 2^k \ell} \text{ so that } |\mu - \mu_k| \leq 2^{-k/2}\}. \end{aligned} \quad (2.72)$$

Clearly, the sets $\mathcal{K}_{1,\ell}$, $\tilde{\mathcal{K}}_{2,\ell}$, and $\mathcal{K}_{2,\ell}$ are compact separable in the weak topology. Also, we take λ , the normalized Lebesgue measure on Ω , and

$$\mathcal{K}_{3,\ell} = \Omega \setminus \partial_{1/\ell} \Omega = \{x \in \Omega : \text{dist}(x, \partial \Omega) \geq 1/\ell\}, \quad (2.73)$$

and for the probabilities (S_n, μ_n, Σ_n) we take $S_n = S$ and $\mu_n = \mu_{\varepsilon_n, u}$ with $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$.

Finally, we introduce the family of mappings $\mathcal{T}_{\Lambda, n} : S \times \Omega \rightarrow (W_0^{1,p}(\Omega) + u) \times BC(S)^*$ defined by

$$\mathcal{T}_{\Lambda, n}(X, x) = (\Pi_\varepsilon(X), \mu_\Lambda(\cdot \mid \tau_{\lfloor x/\varepsilon_n \rfloor}(X))). \quad (2.74)$$

We will consider the sequences $\mathcal{T}_{\Lambda_N, n}$, first in n and then in N , and show that they satisfy a uniform tightness condition. To this end we verify the following bounds.

Lemma 2.14. *There exist fixed constants \tilde{c} and $\tilde{\ell}$ such that, uniformly in Λ ,*

- (a) $\mu_{\varepsilon, u}(\{X \in S : \int_\Omega |\nabla \Pi_\varepsilon(X)(x)|^p d\lambda(x) \geq \ell\}) \leq \exp(-\tilde{c}\ell\varepsilon^{-d})$ and $\int(\int_\Omega (|\nabla \Pi_\varepsilon(X)(x)|^p - \ell)_+ d\lambda(x)) d\mu_{\varepsilon, u}(X) \leq \exp(-\tilde{c}\ell\varepsilon^{-d})$ for any $\ell \geq \tilde{\ell}$.
- (b) $\int_{\mathcal{K}_{3,\ell}} \int (1 - \mathbb{1}_{\mathcal{K}_{2,\ell}})(\mu_\Lambda(\cdot \mid \tau_{\lfloor x/\varepsilon \rfloor}(X))) d\mu_{\varepsilon, u}(X) d\lambda(x) \leq \tilde{c}(1 + \frac{\|u\|_{1,p}}{\ell})^{\tilde{\ell}} \frac{\tilde{\ell}}{\ell}$ whenever $\varepsilon \text{diam } \Lambda < 1/\ell$.

Proof.

(a) is an immediate consequence of the assumption (A1) and Exponential Tightness once we observe that $\{X : \sum_{i \in \Omega_\varepsilon} |\nabla X(i)|^p > \varepsilon^{-d}\ell\} \subset \mathcal{M}_K$ with $K = \frac{c\ell}{|\Omega|}$.

(b) Notice that $\lambda(\partial_{1/\ell}\Omega) \leq C_{\partial} \frac{|\partial\Omega|}{|\Omega|} \frac{1}{\ell}$ in view of the condition (A_{∂}) . On several occasions we will use the DLR condition in the following form: whenever $\Lambda \subset \Omega_{\varepsilon}$ and f, g are measurable cylinder functions on S with g living on $\mathbb{Z}^d \setminus \Lambda$, then

$$\int \mu_{\Lambda}(f \mid X) g(X) d\mu_{\varepsilon, u}(X) = \mu_{\varepsilon, u}(fg). \quad (2.75)$$

Using this (with $g = 1$) and assuming that $\varepsilon \text{diam } \Lambda < 1/\ell$, we have

$$\begin{aligned} \int_{\mathcal{K}_{3, \ell}} \int \mu_{\Lambda}(f \mid \tau_{\lfloor x/\varepsilon \rfloor}(X)) d\mu_{\varepsilon, u}(X) d\lambda(x) &= \\ &= \int_{\mathcal{K}_{3, \ell}} \int \mu_{\tau_{\lfloor x/\varepsilon \rfloor}(\Lambda)}(f \circ \tau_{\lfloor x/\varepsilon \rfloor} \mid X) d\mu_{\varepsilon, u}(X) d\lambda(x) = \int_{\mathcal{K}_{3, \ell}} \mu_{\varepsilon, u}(f \circ \tau_{\lfloor x/\varepsilon \rfloor}) d\lambda(x). \end{aligned} \quad (2.76)$$

Taking $f = \mathbb{1}_{B_{\Lambda_N, 2^{N+k}|\Lambda_N|}^c}^c$, we get

$$\begin{aligned} \int_{\mathcal{K}_{3, \ell}} \int (1 - \mathbb{1}_{\mathcal{K}_{2, \ell}})(\mu_{\Lambda}(\cdot \mid \tau_{\lfloor x/\varepsilon \rfloor}(X))) d\mu_{\varepsilon, u}(X) d\lambda(x) &\leq \\ &\leq \sum_k 2^{k/2} \int_{\mathcal{K}_{3, \ell}} \mu_{\varepsilon, u}(\cup_N B_{\tau_{\lfloor x/\varepsilon \rfloor}(\Lambda_N), 2^{N+k}|\Lambda_N|}^c) d\lambda(x) \leq \\ &\leq \sum_{k, N} 2^{k/2} \int_{\Omega} \int \frac{1}{2^{N+k}|\Lambda_N|} \sum_{i \in \tau_{\lfloor x/\varepsilon \rfloor}(\Lambda_N)} |\nabla X(i)|^p d\mu_{\varepsilon, u}(X) d\lambda(x) \leq \\ &\leq \sum_{k, N} 2^{-N-k/2} \int \frac{\varepsilon^d}{\ell} \sum_{i \in \mathbb{Z}^d} |\nabla X(i)|^p d\mu_{\varepsilon, u}(X) \leq \\ &\leq \sum_{k, N} 2^{-N-k/2} \frac{1}{\ell} [\|u\|_{1, p} + \tilde{\ell} + \sum_{\ell \geq 0} 2^{\ell+1} \tilde{\ell} \mu_{\varepsilon, u}(\{X : \varepsilon^d \sum_{i \in \Omega_{\varepsilon}} |\nabla X(i)|^p \in [2^{\ell} \tilde{\ell}, 2^{\ell+1} \tilde{\ell}]\})] \leq \\ &\leq \sum_{k, N} 2^{-N-k/2} \frac{\tilde{\ell}}{\ell} [\frac{\|u\|_{1, p}}{\tilde{\ell}} + 1 + \sum_{\ell \geq 0} 2^{\ell+1} \exp(-\tilde{c} 2^{\ell} \tilde{\ell} \varepsilon^{-d})] \leq \\ &\leq \frac{\tilde{\ell}}{\ell} \sum_{k, N} 2^{-N-k/2} (\frac{\|u\|_{1, p}}{\tilde{\ell}} + 1 + 2 \sum_{\ell} 2^{-\ell}) \leq 2\sqrt{2}(\sqrt{2} + 1)(5 + \frac{\|u\|_{1, p}}{\tilde{\ell}}) \frac{\tilde{\ell}}{\ell} \end{aligned} \quad (2.77)$$

once $\tilde{\ell}$ is large enough (and for ε small enough) so that $t^2 e^{-\tilde{c} \tilde{\ell} \varepsilon^{-d} t} \leq 1$ for any $t \geq 0$. \square

Applying now Lemma 2.13, we get the claim (1.43) for any $\varphi \in BC((W_0^{1, p}(\Omega) + u) \times BC(S)_+^* \times \Omega)$. To extend (1.43) to a more general class of test functions, we will use the following Lemma.

Lemma 2.15. *Let $\nu_{\tilde{\Lambda},\varepsilon}(\varphi) = \mu_{\varepsilon,u}(\int_{\Omega} \varphi(\Pi_{\varepsilon}(X), \mu_{\tilde{\Lambda}}(\cdot|\tau_{[x/\varepsilon]}(X)), x) d\lambda(x))$. Consider the weak closure \mathcal{M} of the set $\{\nu_{\tilde{\Lambda},\varepsilon} : \varepsilon \text{diam}\tilde{\Lambda} < \rho\}$. Further, let*

$$\psi : (W_0^{1,p}(\Omega) + u) \times BC(S)_+^* \times \Omega \rightarrow \mathbb{R} \quad (2.78)$$

be weakly continuous and such that for any δ it fulfills the growth condition

$$|\psi(v, \mu, x)| \leq \eta(x) \exp\{c_{\psi} \|\nabla v\|_p^p\} \mu(\delta g_{\Lambda} + C(\delta)) \quad (2.79)$$

with fixed Λ , $g_{\Lambda}(X) = \sum_{i \in \Lambda} |\nabla X(i)|^p$, $\eta \in C_0(\Omega)$ such that $\text{dist}(\text{supp } \eta, \partial\Omega) > \rho$, and a constant c_{ψ} depending only on ψ . Then, if for some $\nu_n, \nu \in \mathcal{M}$ we have $\nu_n(\varphi) \rightarrow \nu(\varphi)$ for all $\varphi \in BC((W_0^{1,p}(\Omega) + u) \times BC(S)_+^ \times \Omega)$, then also $\nu_n(\psi) \rightarrow \nu(\psi)$.*

Remark 2.16. *Notice that, due to Poincaré inequality, the set of test functions above is the same if we replace $\|\nabla v\|_p^p$ in (2.79) by $\|v\|_{1,p}^p$.*

Proof. Since $\psi \min(1, \frac{\ell}{|\psi|}) \in BC((W_0^{1,p}(\Omega) + u) \times BC(S)_+^* \times \Omega)$, it is enough to prove that

$$\nu((\psi - \ell)_+) \leq \omega(\ell) \text{ with } \lim_{\ell \rightarrow \infty} \omega(\ell) = 0 \quad (2.80)$$

uniformly in $\nu \in \mathcal{M}$. For any $k > 0$, let us decompose ψ as follows,

$$\psi = \psi \mathbb{1}_{\|\nabla v\|_p^p \leq k} + \psi \mathbb{1}_{\|\nabla v\|_p^p > k}. \quad (2.81)$$

For the first term, we notice that

$$|\psi(v, \mu, x)| \mathbb{1}_{\|\nabla v\|_p^p \leq k}(v) \leq \eta(x) e^{c_{\psi}k} (C(\delta) + \delta \mu(g_{\Lambda})), \quad (2.82)$$

implies

$$(\psi(v, \mu, x) - \ell)_+ \mathbb{1}_{\|\nabla v\|_p^p \leq k}(v) \leq \delta \eta(x) e^{c_{\psi}k} \mu(g_{\Lambda}) \quad (2.83)$$

once $\ell > \ell(k, \delta) = \|\eta\| e^{c_{\psi}k} C(\delta)$ and thus, for $\nu = \nu_{\tilde{\Lambda},\varepsilon}$ with $\text{dist}(\text{supp } \eta, \partial\Omega) > \varepsilon \text{diam}\tilde{\Lambda}$, we get

$$\begin{aligned} \nu((\psi - \ell)_+ \mathbb{1}_{\|\nabla v\|_p^p \leq k}) &\leq \delta e^{c_{\psi}k} \int \left(\int \eta(x) \mu_{\tilde{\Lambda}}(g_{\Lambda}|\tau_{[x/\varepsilon]}(X)) d\lambda(x) \right) d\mu_{\varepsilon,u}(X) = \\ &= \delta e^{c_{\psi}k} \int \eta(x) \left(\int g_{\tau_{[x/\varepsilon]} \Lambda}(X) d\mu_{\varepsilon,u}(X) \right) d\lambda(x) \leq \delta e^{c_{\psi}k} \|\eta\| |\Lambda| \tilde{C} \end{aligned} \quad (2.84)$$

with \tilde{C} denoting the bound on $\int \|\nabla \Pi_{\varepsilon} X\|_p^p d\mu_{\varepsilon,u}(X)$ (cf. Lemma 2.14 (a)). Here, we first used that $\int \mu_{\tilde{\Lambda}}(g_{\Lambda}|\tau_{[x/\varepsilon]}(X)) d\mu_{\varepsilon,u}(X) = \mu_{\varepsilon,u}(g_{\Lambda} \circ \tau_{[x/\varepsilon]})$ similarly as in (2.76) and then bounded $\int \eta(x) g_{\tau_{[x/\varepsilon]} \Lambda}(X) d\lambda(x) \leq \|\eta\| |\Lambda| \varepsilon^d \sum_{i \in \Omega_{\varepsilon}} |\nabla X(i)|^p$.

Further, we will slice $\psi \mathbb{1}_{\|\nabla v\|_p^p > k} = \sum_{j \geq 0} \psi_{2^j k}$ with the functions $\psi_n = \psi \mathbb{1}_{n < \|\nabla v\|_p^p \leq 2n}$ satisfying the bound (with $\delta = 1$),

$$|\psi_n(v, \mu, x)| \leq \eta(x) e^{2c_{\psi}n} (\mu(g_{\Lambda}) + C(1)) \mathbb{1}_{n < \|\nabla v\|_p^p \leq 2n}(v). \quad (2.85)$$

In preparation for the evaluation of $\nu(\psi_n) = \nu_{\tilde{\Lambda}, \varepsilon}(\psi_n)$, we use $(\tilde{\Lambda} \cup \Lambda)_1$ to denote the 1-neighbourhood of $\tilde{\Lambda} \cup \Lambda$ and for any $X, Y \in (\mathbb{R}^m)^{\mathbb{Z}^d}$ we bound

$$\begin{aligned} & g_{\Lambda}(Y) \mathbb{1}_{n < \|\nabla \Pi_{\varepsilon}(X)\|_p^p \leq 2n}(X) \leq \\ & \leq (g_{\Lambda}(Y) - g_{(\tilde{\Lambda} \cup \Lambda)_1}(\tau_{\lfloor x/\varepsilon \rfloor}(X)))_+ \mathbb{1}_{n < \|\nabla \Pi_{\varepsilon}(X)\|_p^p}(X) + g_{(\tilde{\Lambda} \cup \Lambda)_1}(\tau_{\lfloor x/\varepsilon \rfloor}(X)) \mathbb{1}_{n < \|\nabla \Pi_{\varepsilon}(X)\|_p^p \leq 2n}(X) \leq \\ & \leq g_{\Lambda}(Y) \mathbb{1}_{n/2 < \|\nabla \Pi_{\varepsilon}(X)\|_p^p - \varepsilon^d g_{\tau_{\lfloor x/\varepsilon \rfloor}(\tilde{\Lambda} \cup \Lambda)_1}(X)}(X) + (g_{\Lambda}(Y) - \varepsilon^{-d} n/2)_+ + \\ & \quad + g_{(\tilde{\Lambda} \cup \Lambda)_1}(\tau_{\lfloor x/\varepsilon \rfloor}(X)) \mathbb{1}_{n < \|\nabla \Pi_{\varepsilon}(X)\|_p^p \leq 2n}(X). \end{aligned} \quad (2.86)$$

Notice that $\|\nabla \Pi_{\varepsilon}(X)\|_p^p - \varepsilon^d g_{\tau_{\lfloor x/\varepsilon \rfloor}(\tilde{\Lambda} \cup \Lambda)_1}(X) = \varepsilon^d g_{\Omega_{\varepsilon} \setminus \tau_{\lfloor x/\varepsilon \rfloor}(\tilde{\Lambda} \cup \Lambda)_1}(X)$ and thus the right hand side above actually does not depend on $X(i), i \in \tau_{\lfloor x/\varepsilon \rfloor}(\tilde{\Lambda} \cup \Lambda)$. As a result, using (2.75) we get

$$\begin{aligned} & \int \mu_{\tilde{\Lambda}}(g_{\Lambda} | \tau_{\lfloor x/\varepsilon \rfloor}(X)) \mathbb{1}_{n < \|\nabla \Pi_{\varepsilon}(X)\|_p^p \leq 2n} d\mu_{\varepsilon, u}(X) \leq \mu_{\varepsilon, u}((g_{\Lambda} \circ \tau_{\lfloor x/\varepsilon \rfloor}) \mathbb{1}_{n/2 < \|\nabla \Pi_{\varepsilon}(\cdot)\|_p^p}) + \\ & + \mu_{\varepsilon, u}((g_{\Lambda} \circ \tau_{\lfloor x/\varepsilon \rfloor} - \varepsilon^{-d} n/2)_+) + \int g_{(\tilde{\Lambda} \cup \Lambda)_1}(\tau_{\lfloor x/\varepsilon \rfloor}(X)) \mathbb{1}_{n < \|\nabla \Pi_{\varepsilon}(X)\|_p^p \leq 2n}(X) d\mu_{\varepsilon, u}(X) \end{aligned} \quad (2.87)$$

Hence, once $\varepsilon \text{diam} \tilde{\Lambda} < \text{dist}(\text{supp } \eta, \partial \Omega)$, we bound $\nu(\psi_n)$, up to a prefactor $e^{2c_{\psi_n} n} \|\eta\|$, by

$$\int (C(1) + 2n |\tilde{\Lambda} \cup \Lambda| \mathbb{1}_{n < \|\nabla \Pi_{\varepsilon}(X)\|_p^p}(X) + 2 |\Lambda| \|\nabla \Pi_{\varepsilon}(X)\|_p^p \mathbb{1}_{\|\nabla \Pi_{\varepsilon}(X)\|_p^p + \geq n/2}(X)) d\mu_{\varepsilon, u}(X). \quad (2.88)$$

Thus, for n large,

$$\nu(\psi_n) \leq e^{2c_{\psi_n} n} \|\eta\| ((C(1) + |\tilde{\Lambda} \cup \Lambda| 2n) e^{-\tilde{c} n \varepsilon^{-d}} + \bar{C} |\Lambda| e^{-\tilde{c} n/2 \varepsilon^{-d}}) \quad (2.89)$$

implying the claim. \square

To show that the family $\nu_{x, v}$ of Young measures has support in the set of Gibbs measures, observe that $\mu \in \mathcal{G}$ iff

$$\mu(\mu_{\Lambda}(f|\cdot)) - \mu(f) = \varphi_{\Lambda, f}(\mu) = 0 \quad (2.90)$$

for any finite Λ and any cylinder function f living in Λ . Noticing that $\varphi_{\Lambda, f}$ is a bounded test function, $\varphi_{\Lambda, f} \in BC((W_0^{1,p}(\Omega) + u) \times BC(S)_+^* \times \Omega)$ we just have to verify that

$$\begin{aligned} \nu_{\tilde{\Lambda}, \varepsilon}(\varphi_{\Lambda, f}) &= \int \int_{\Omega} \varphi_{\Lambda, f}(\mu_{\tilde{\Lambda}}(\cdot | \tau_{\lfloor x/\varepsilon \rfloor}(X))) d\lambda(x) d\mu_{\varepsilon, u}(X) = \\ &= \int \int_{\Omega} (\mu_{\tilde{\Lambda}}(\mu_{\Lambda}(f|\cdot) | \tau_{\lfloor x/\varepsilon \rfloor}(X)) - \mu_{\tilde{\Lambda}}(f | \tau_{\lfloor x/\varepsilon \rfloor}(X))) d\lambda(x) d\mu_{\varepsilon, u}(X) = 0 \end{aligned} \quad (2.91)$$

since $\mu_{\tilde{\Lambda}}(\mu_{\Lambda}(f|\cdot)|\tau_{\lfloor x/\varepsilon \rfloor}(X)) = \mu_{\tilde{\Lambda}}(f|\tau_{\lfloor x/\varepsilon \rfloor}(X))$ once $\Lambda \subset \tilde{\Lambda}$.

To show that $\int \mathbb{E}_{\mu}(\nabla X(0)) d\nu_{x,v}(\mu) = \nabla v(x)$, we use the test function

$$\varphi(v, \mu, x) = \eta(x) \mathbb{E}_{\mu}(\nabla X(0) - \nabla v(x)) \quad (2.92)$$

with $\eta \in C^1(\overline{\Omega})^{m \times d}$ and observe that

$$\begin{aligned} & \left| \int \int \varphi(\Pi_{\varepsilon}(X), \mu_{\tilde{\Lambda}}(\cdot|\tau_{\lfloor x/\varepsilon \rfloor}(X)), x) d\lambda(x) d\mu_{\varepsilon,u}(X) \right| \leq \\ & \leq \left| \int \eta(x) \int [\nabla X(\lfloor x/\varepsilon \rfloor) - \nabla(\Pi_{\varepsilon}(X))(x)] d\mu_{\varepsilon,u}(X) d\lambda(x) \right| \leq \\ & \leq \text{const } \varepsilon \|\nabla \eta\|_{\infty} \mu_{\varepsilon,u}(\|\nabla \Pi_{\varepsilon}(X)\|_1). \end{aligned} \quad (2.93)$$

The last estimate is valid for the linear interpolation; for a more general case, $\|\nabla \eta\|_{\infty}$ will be replaced by $\|\nabla^2 \eta\|_{\infty}$.

Finally, to show that $\nu_{v,x}(|\nabla X(0)|^p) < \infty$, we can use the function $\varphi_k(v, \mu, x) = \mu(\min(k, |\nabla X(0)|^p))$ as a test function yielding the bound uniform in k . \square

APPENDIX A. TECHNICAL LEMMAS

We begin with a technical Lemma that will be useful on several occasions.

Lemma A.1.

Let $a > 0$ and $\Lambda \subset \Omega_{\varepsilon}$ be connected (when viewed as a subgraph of \mathbb{Z}^d with the set of edges consisting of all pairs of nearest neighbours (i, j) , $|i - j| = 1$). Then:

a) We have

$$\int \mathbb{1}_{\{j\},y}(X) \exp\left(-a \sum_{i \in \Lambda} |\nabla X(i)|^p\right) \prod_{i \in \Lambda} dX(i) \leq \omega(m) (a^{-m/p} c(p, m))^{| \Lambda | - 1}, \quad (A.1)$$

where $j \in \Lambda$ and $\mathbb{1}_{\{j\},y}$ is the indicator of the set $\{X \in (\mathbb{R}^m)^{\Lambda} \mid |X(j) - y| < 1\}$ and $\omega(m)$ is the volume of the unit ball in \mathbb{R}^m .

b) For any $v \in L^r(\Omega, \mathbb{R}^m)$ and ε sufficiently small,

$$\int_{\mathcal{N}_{\Lambda,r}(v,\kappa)} \exp\left(-a \sum_{i \in \Lambda} |\nabla X(i)|^p\right) \prod_{i \in \Lambda} dX(i) \leq \vartheta |\Lambda|^{1+\frac{m}{d}} (a^{-m/p} c(p, m))^{| \Lambda | - 1}, \quad (A.2)$$

where $\vartheta = \omega(m) \kappa^m$ and $c(p, m) = \int_{\mathbb{R}^m} \exp(-|\xi|^p) d\xi$.

Proof. a) Consider a tree t rooted at the site j . Then

$$-\sum_{i \in \Lambda} |\nabla X(i)|^p \leq -\sum_{\{i,j\} \in t} |X(i) - X(j)|^p. \quad (A.3)$$

and thus

$$\int \mathbb{1}_{\{j\},y}(X) \exp\left(-a \sum_{i \in \Lambda} |\nabla X(i)|^p\right) \prod_{i \in \Lambda} dX(i) \leq \omega(m) \cdot \left(a^{-m/p} c(p, m)\right)^{| \Lambda | - 1} \quad (A.4)$$

b) The set Λ is connected and can be covered by a spanning tree t implying (A.3). Further, we clearly have $\mathcal{N}_{\Lambda,r}(v, \kappa) \subset \cup_{j \in \Omega_\varepsilon} \mathcal{N}_r^{(j)}(v, \kappa)$ with

$$\mathcal{N}_r^{(j)}(v, \kappa) = \{X : \Lambda \rightarrow \mathbb{R}^m \mid |X(j) - X_{v,\varepsilon}(j)| < \kappa |\Lambda|^{\frac{1}{d}}\}. \quad (\text{A.5})$$

Considering the tree t as rooted at j , we get

$$\int_{\mathcal{N}_r^{(j)}(u, \kappa)} \exp\left(-a \sum_{i \in \Lambda} |\nabla X(i)|^p\right) \prod_{i \in \Lambda} dX(i) \leq \omega(m) \kappa^m |\Lambda|^{\frac{m}{d}} \left(a^{-m/p} c(p, m)\right)^{|\Lambda|-1} \quad (\text{A.6})$$

implying the claim with the help of a). \square

Remark A.2. An immediate consequence of Lemma A.1 a), under the assumption (A1), is the bound

$$Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon,r}(u, \kappa)) \leq \vartheta |\Omega_\varepsilon|^{1+\frac{m}{d}} \left(c^{-m/p} c(p, m)\right)^{|\Omega_\varepsilon|-1}. \quad (\text{A.7})$$

Proof of Exponential Tightness Lemma.

The bound $H_{\Omega_\varepsilon}(X) > K |\Omega_\varepsilon|$ and (A1) implies that

$$-H_{\Omega_\varepsilon}(X) < -\frac{1}{2} K |\Omega_\varepsilon| - \frac{1}{2} c \sum_{\substack{j \in \Omega_\varepsilon \\ \tau_j(A) \subset \Omega_\varepsilon}} |\nabla X(j)|^p \quad (\text{A.8})$$

for all $X \in \mathcal{M}_K$. Hence,

$$\begin{aligned} Z_{\Omega_\varepsilon}(\mathcal{M}_K \cap \mathcal{N}_{\Omega_\varepsilon, R_0, \infty}(X_{u, \varepsilon})) &\leq \\ &\leq \exp\left(-\frac{1}{2} K |\Omega_\varepsilon|\right) \int_{\mathcal{N}_{\Omega_\varepsilon, R_0, \infty}(X_{u, \varepsilon})} \exp\left(-\frac{1}{2} c \sum_{\substack{j \in \Omega_\varepsilon \\ \tau_j(A) \subset \Omega_\varepsilon}} |\nabla X(j)|^p\right) \prod_{i \in \Omega_\varepsilon} dX(i) \end{aligned} \quad (\text{A.9})$$

Consider the set $\Omega_\varepsilon^0 = \{j \in \mathbb{Z}^d \mid \tau_j(A) \subset \Omega_\varepsilon\}$. For sufficiently small ε , the set Ω_ε^0 is connected, $|\Omega_\varepsilon \setminus \Omega_\varepsilon^0| \leq C_{\partial} R_0 \varepsilon^{-d+1} |\partial \Omega|$ and $\Omega_\varepsilon^0 \cap S_{R_0}(\Omega_\varepsilon) \neq \emptyset$. Thus, $X \in \mathcal{N}_{\Omega_\varepsilon, R_0, \infty}(X_{u, \varepsilon})$ implies that $|X(j) - X_{u, \varepsilon}(j)| \leq 1$ for every $j \in \Omega_\varepsilon^0 \cap S_{R_0}(\Omega_\varepsilon)$. Hence, using Lemma A.1 a) to bound the integral on the right hand side, we get

$$\begin{aligned} \int_{\mathcal{N}_{\Omega_\varepsilon, R_0, \infty}(X_{u, \varepsilon})} \exp\left(-\frac{1}{2} c \sum_{\substack{j \in \Omega_\varepsilon \\ \tau_j(A) \subset \Omega_\varepsilon}} |\nabla X(j)|^p\right) \prod_{i \in \Omega_\varepsilon} dX(i) &\leq \\ &\leq \omega(m) \left[\left(\frac{2}{c}\right)^{m/p} c(p, m)\right]^{|\Omega_\varepsilon|} \left(\frac{\varepsilon_0}{\varepsilon}\right)^{m(1+\frac{d}{r}) C_{\partial} R_0 \varepsilon^{-d+1} |\partial \Omega|} \end{aligned} \quad (\text{A.10})$$

once $\varepsilon \leq \varepsilon_0$ with $\varepsilon_0 = \varepsilon_0(\kappa, r) = \left(\kappa \left(\frac{2}{c}\right)^{-1/p} c(p, m)^{-1/m}\right)^{\frac{r}{d+r}} |\Omega|^{\frac{1}{d}}$. Thus

$$Z_{\Omega_\varepsilon}(\mathcal{M}_K \cap \mathcal{N}_{\Omega_\varepsilon, r}(u, \kappa)) \leq \exp\left(-\frac{1}{2} K |\Omega_\varepsilon|\right) \left[2 \left(\frac{4}{c}\right)^{m/p} c(p, m)\right]^{|\Omega_\varepsilon|}. \quad (\text{A.11})$$

Here, we also used the bound $\exp\left(-\frac{1}{2}K|\Omega_\varepsilon|\right)\left(\frac{\varepsilon_0}{\varepsilon}\right)^{m(1+\frac{d}{r})C_\partial R_0\varepsilon^{-d+1}|\partial\Omega|} \leq 1$ valid whenever

$$K > \varepsilon|\log(\frac{\varepsilon}{\varepsilon_0})| m(1 + \frac{d}{r})4C_\partial R_0 \frac{|\partial\Omega|}{|\Omega|}. \quad (\text{A.12})$$

For the second bound we derive

$$Z_{\Omega_\varepsilon}(\mathcal{M}_K \cap \mathcal{N}_{\Omega_\varepsilon, r}(u, \kappa)) \leq \exp\left(-\frac{1}{2}K|\Omega_\varepsilon|\right) \int_{\mathcal{N}_{\Omega_\varepsilon, r}(u, \kappa)} \exp\left(-\frac{1}{2}c \sum_{\substack{j \in \Omega_\varepsilon \\ \tau_j(A) \subset \Omega_\varepsilon}} |\nabla X(j)|^p\right) \prod_{i \in \Omega_\varepsilon} dX(i) \quad (\text{A.13})$$

in a similar way, using the fact that $X \in \mathcal{N}_{\Omega_\varepsilon, r}(u, \kappa)$ implies that $|X(i) - \frac{1}{\varepsilon}u(\varepsilon i)| \leq \kappa|\Omega_\varepsilon|^{\frac{1}{r}+\frac{1}{d}}$ for every $i \in \Omega_\varepsilon$ and applying Lemma A.1 b) to bound the integral on the right hand side. We also assumed that ε is sufficiently small so that $\vartheta|\Omega_\varepsilon|^{1+\frac{m}{d}} \leq 2^{|\Omega_\varepsilon|}$. \square

Proof of Interpolation Lemma.

Fixing parameters $\eta > 0$ and $N \in \mathbb{N}$, we slice the strip $(\partial_\eta \Omega)_\varepsilon$ into strips of width $\frac{\eta}{\varepsilon N}$. In particular, we pick up $R = R(\varepsilon)$ so that $R > 2R_0$ and $\eta = N\varepsilon R$ and partition the set $\mathcal{N}_{\Omega_\varepsilon, r}(u, \kappa) = \cup_{\ell=1}^{N-1} \widehat{\mathcal{N}}_r^{(\ell)}(u, \kappa)$ with

$$\widehat{\mathcal{N}}_r^{(k)}(u, \kappa) = \{X \in \mathcal{N}_{\Omega_\varepsilon, r}(u, \kappa) \mid \sum_{j \in S_{R_0+kR} \setminus S_{R_0+(k-1)R}} U(X_{\tau_j(A)}) \leq \frac{1}{N-1} H_{\Omega_\varepsilon}(X)\}, \quad (\text{A.14})$$

where $S_{R_0+kR} = S_{R_0+kR}(\Omega_\varepsilon)$ is the strip $S_{R_0+kR} = \{i \in \Omega_\varepsilon : \varepsilon^{-1} \text{dist}(\varepsilon i, \Omega^c) \leq R_0 + kR\}$. To see that, indeed, $\mathcal{N}_{\Omega_\varepsilon, r}(u, \kappa) \subset \cup_{k=1}^{N-1} \widehat{\mathcal{N}}_r^{(k)}(u, \kappa)$, it suffices to show that any X from the set $\mathcal{N}_{\Omega_\varepsilon, r}(u, \kappa) \setminus \cup_{k=1}^{N-1} \widehat{\mathcal{N}}_r^{(k)}(u, \kappa)$ would necessarily satisfy $H_{S_{NR}}(X) > H_{\Omega_\varepsilon}(X)$ which is contradiction due to nonnegativity of $U(X_A)$. Further, introducing the function

$$\Theta_k(i) = \min(1, R^{-1}(\varepsilon^{-1} \text{dist}(\varepsilon i, \Omega^c) - R_0 - (k-1)R)_+) \quad (\text{A.15})$$

on Ω_ε interpolating between 1 on $\Omega_\varepsilon \setminus S_{R_0+kR}$ and 0 on $S_{R_0+(k-1)R}$, we define, for any $X \in (\mathbb{R}^m)^{\Omega_\varepsilon}$ and $Y \in (\mathbb{R}^m)^{S_{NR}}$ the function $T_k(X, Y) \in (\mathbb{R}^m)^{\Omega_\varepsilon}$ by

$$T_k(X, Y)(i) = \Theta_k(i)X(i) + (1 - \Theta_k(i))Y(i). \quad (\text{A.16})$$

It is interpolating between $T_k(X, Y)(i) = X(i)$ on $\Omega_\varepsilon \setminus S_{R_0+kR}$ and $T_k(X, Y)(i) = Y(i)$ on $S_{R_0+(k-1)R}$.

Let $Z \in \mathcal{N}_{\Omega_\varepsilon, r}(u, \kappa)$ and consider $X \in \widehat{\mathcal{N}}_r^{(k)}(u, \kappa)$ and $Y \in \mathcal{N}_{S_{NR}, \infty}(Z)$. For the completeness of the argument, let us first show that $T_k(X, Y) \in \mathcal{N}_{\Omega_\varepsilon, r}(u, 3\kappa) \cap \mathcal{N}_{\Omega_\varepsilon, R_0, \infty}(Z)$ for each $1 \leq k \leq N-1$. Indeed, extending Y to Ω_ε by taking $Y(i) =$

$Z(i)$ on $\Omega_\varepsilon \setminus S_{NR}$ and using that $X, Z \in \mathcal{N}_{\Omega_\varepsilon, r}(u, \kappa)$, we get

$$\begin{aligned} & \|T_k(X, Y) - X_{u, \varepsilon}\|_{\ell^r(\Omega_\varepsilon)} \leq \\ & \leq \|\Theta_k(X - X_{u, \varepsilon})\|_{\ell^r(\Omega_\varepsilon)} + \|(1 - \Theta_k)(Y - Z)\|_{\ell^r(\Omega_\varepsilon)} + \|(1 - \Theta_k)(Z - X_{u, \varepsilon})\|_{\ell^r(\Omega_\varepsilon)} \leq \\ & \leq \left(2\kappa + |S_{NR}|^{\frac{1}{r}} |\Omega_\varepsilon|^{-\frac{1}{r} - \frac{1}{d}}\right) |\Omega_\varepsilon|^{\frac{1}{r} + \frac{1}{d}} \leq 3\kappa |\Omega_\varepsilon|^{\frac{1}{r} + \frac{1}{d}}. \quad (\text{A.17}) \end{aligned}$$

Here, we first bounded

$$|S_{NR}|^{\frac{1}{r}} |\Omega_\varepsilon|^{-\frac{1}{r} - \frac{1}{d}} \leq (C_\partial NR |\partial\Omega| \varepsilon^{-(d-1)})^{\frac{1}{r}} |\Omega|^{-\frac{1}{r} - \frac{1}{d}} \varepsilon^{1 + \frac{d}{r}} = \frac{\varepsilon}{|\Omega|^{1/d}} \eta^{\frac{1}{r}} \left(\frac{C_\partial |\partial\Omega|}{|\Omega|}\right)^{\frac{1}{r}} \quad (\text{A.18})$$

with $\eta = NR\varepsilon$ and assumed that ε is sufficiently small to assure that, with fixed η , the right hand side above does not exceed κ .

The main idea of the proof is to introduce a new integral quantity that serves as an upper bound to the left hand side of (2.10) and, in the same time, as a lower bound of its right hand side. To be more precise, for verification of an inequality of the form (2.10) with the integral on the left hand side restricted to $\widehat{\mathcal{N}}_r^{(\ell)}(u, \kappa)$, we “double the variables” and introduce the following integral over $(\mathbb{R}^m)^{\Omega_\varepsilon} \times (\mathbb{R}^m)^{S_{R_0+kR}}$,

$$I_k = \int_{\widehat{\mathcal{N}}_r^{(\ell)}(u, \kappa) \times \mathcal{N}_{S_{R_0+kR}, \infty}(Z)} \exp(-H_{\Omega_\varepsilon}(T_k(X, Y)) - a \sum_{j \in S_{R_0+kR}} |\nabla X(j)|^p) \prod_{i \in \Omega_\varepsilon} dX(i) \prod_{j \in S_{R_0+kR}} dY(j). \quad (\text{A.19})$$

First, let us attend to the *lower bound* on I_k . For the terms $U(T_k(X, Y)_{\tau_j(A)})$ contributing to $H_{\Omega_\varepsilon}(T_k(X, Y))$ we consider 3 cases:

- (i) If $\tau_j(A) \cap S_{R_0+kR} = \emptyset$, then $U(T_k(X, Y)_{\tau_j(A)}) = U(X_{\tau_j(A)})$.
- (ii) If $\tau_j(A) \cap (S_{R_0+kR} \setminus S_{R_0+(k-1)R}) \neq \emptyset$, then, by assumption (A2),

$$U(T_k(X, Y)_{\tau_j(A)}) \leq C(1 + U(X_{\tau_j(A)}) + U(Y_{\tau_j(A)}) + \sum_{i \in \tau_j(A)} |\Theta_k(i) - \Theta_k(j)|^r |X(i) - Y(i)|^r). \quad (\text{A.20})$$

In this inequality we used the fact that

$$T_k(X, Y)(i) = \Theta_k(j)X(i) + (1 - \Theta_k(j))Y(i) + (\Theta_k(i) - \Theta_k(j))(X(i) - Y(i)). \quad (\text{A.21})$$

Again, for $i \notin S_{NR}$, we have $Y(i) = Z(i)$.

- (iii) If $\tau_j(A) \subset S_{R_0+(k-1)R}$, then $U(T_k(X, Y)_{\tau_j(A)}) = U(Y_{\tau_j(A)})$.

The terms $a|\nabla X(j)|^p$ in the integrand of I_k are, according to (A1), bounded as

$$a|\nabla X(j)|^p \leq U(X_{\tau_j(A)}) \quad (\text{A.22})$$

for any $j \in S_{R_0+kR}$ and $a \leq c$. As a result, using also the assumption (A2) in the form (1.11),

$$U(Y_{\tau_j(A)}) \leq C(1 + U(Z_{\tau_j(A)}) + \sum_{i \in \tau_j(A)} |Y(i) - Z(i)|^r) \leq C(1 + U(Z_{\tau_j(A)}) + R_0^d), \quad (\text{A.23})$$

we are getting the following bounds for the terms

$$L_j = U(T_k(X, Y)_{\tau_j(A)}) + a \mathbb{1}_{j \in S_{R_0+kR}} |\nabla X(j)|^p \quad (\text{A.24})$$

in the 3 cases as above:

- (i) $L_j \leq U(X_{\tau_j(A)})$,
- (ii) $L_j \leq (1+C)U(X_{\tau_j(A)}) + C^2(1 + U(Z_{\tau_j(A)}) + R_0^d) + C +$
 $+ C \sum_{i \in \tau_j(A)} |\Theta_k(i) - \Theta_k(j)|^r |X(i) - Y(i)|^r$,

and

- (iii) $L_j \leq U(X_{\tau_j(A)}) + C(1 + U(Z_{\tau_j(A)}) + R_0^d)$.

Then, for any $X \in \widehat{\mathcal{N}}_r^{(\ell)}(u, \kappa) \setminus \mathcal{M}_K$ and with $a = c$ (and assuming that $C \geq 1$), we have

$$\begin{aligned} H_{\Omega_\varepsilon}(T_k(X, Y)) + c \sum_{i \in S_{R_0+kR}} |\nabla X(i)|^p &\leq H_{\Omega_\varepsilon}(X) + C \frac{K}{N} |\Omega| \varepsilon^{-d} + \\ &+ C^2 \sum_{\substack{j \in S_{NR} \\ \tau_j(A) \subset \Omega_\varepsilon}} U(Z_{\tau_j(A)}) + \tilde{c} |S_{NR}| + C R_0^{d+r} \left(\frac{N}{\eta}\right)^r (2\kappa)^r |\Omega|^{1+\frac{r}{d}} \varepsilon^{-d} \end{aligned} \quad (\text{A.25})$$

with $\tilde{c} = C^2(1+C+R_0^d)+C$. Here, for the last term, we used the following estimate with the \sum_j taken over all $j : \tau_j(A) \cap (S_{R_0+kR} \setminus S_{R_0+(k-1)R}) \neq \emptyset$,

$$\begin{aligned} &\sum_j \sum_{i \in \tau_j(A)} |\Theta_k(i) - \Theta_k(j)|^r |X(i) - Y(i)|^r \leq \\ &\leq \left(\frac{R_0}{R}\right)^r R_0^d \sum_{i \in S_{2R_0+kR} \setminus S_{(k-1)R}} |X(i) - Y(i)|^r \leq R_0^{d+r} R^{-r} (\|X - Z\|_{\ell^r(\Omega_\varepsilon)} + \|Y - Z\|_{\ell^r(S_{NR})})^r \leq \\ &\leq R_0^{d+r} R^{-r} (2\kappa |\Omega_\varepsilon|^{\frac{1}{r}+\frac{1}{d}})^r = R_0^{d+r} \left(\frac{N}{\eta}\right)^r \varepsilon^{-d} (2\kappa)^r |\Omega|^{1+\frac{r}{d}}. \end{aligned} \quad (\text{A.26})$$

To get this, we first used that $|\Theta_k(i) - \Theta_k(j)| \leq \frac{R_0}{R}$ for any $i \in \tau_j(A)$ since $\text{diam} A < R_0$ and then applied the bound from (A.17) assuming that ε is sufficiently small (in dependence on κ and η). As a result, we get

$$H_{\Omega_\varepsilon}(T_k(X, Y)) + c \sum_{i \in S_{R_0+kR}} |\nabla X(i)|^p \leq H_{\Omega_\varepsilon}(X) + \tilde{C} \left(\left(\frac{K}{N} + \eta + \left(\frac{N\kappa}{\eta}\right)^r\right) \varepsilon^{-d} + \sum_{\substack{j \in S_{NR} \\ \tau_j(A) \subset \Omega_\varepsilon}} U(Z_{\tau_j(A)}) \right) \quad (\text{A.27})$$

with

$$\tilde{C} = \max(C|\Omega|, \tilde{c}C_\partial|\partial\Omega|, R_0^{d+r}2^r|\Omega|^{1+\frac{r}{d}}, C^2). \quad (\text{A.28})$$

Thus, finally,

$$Z_{\Omega_\varepsilon}(\widehat{\mathcal{N}}_r^{(\ell)}(u, \kappa) \setminus \mathcal{M}_K) \exp\left\{-\tilde{C}\left(\left(\frac{K}{N} + \eta + \left(\frac{N\kappa}{\eta}\right)^r\right)\varepsilon^{-d} + \sum_{\substack{j \in S_{NR} \\ \tau_j(A) \subset \Omega_\varepsilon}} U(Z_{\tau_j(A)})\right)\right\} \omega(m)^{|S_{R_0+kR}|} \leq I_k. \quad (\text{A.29})$$

For the *upper bound* of the integral I_k , we use the substitution defined as identity on $(\mathbb{R}^m)^{\Omega_\varepsilon \setminus S_{R_0+kR}}$, and, on the remaining $(\mathbb{R}^m)^{S_{R_0+kR}} \times (\mathbb{R}^m)^{S_{R_0+kR}}$, pointwise by the mapping $\Phi_i : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^m \times \mathbb{R}^m$ introduced by

$$\Phi_i(\xi, \zeta) = (\Theta_k(i)\xi + (1 - \Theta_k(i))\zeta, \zeta) \text{ for } i \in S_{R_0+kR} \setminus S_{R_0+(k-1/2)R} \quad (\text{A.30})$$

and by

$$\Phi_i(\xi, \zeta) = (\Theta_k(i)\xi + (1 - \Theta_k(i))\zeta, \xi) \text{ for } i \in S_{R_0+(k-1/2)R}. \quad (\text{A.31})$$

Notice that

$$|\det D\Phi_i|^{-1} \leq 2^m \mathbb{1}_{i \in S_{R_0+kR} \setminus S_{R_0+(k-1)R}} + \mathbb{1}_{i \in S_{R_0+(k-1)R}}. \quad (\text{A.32})$$

Since $T_k(X, Y) \in \mathcal{N}_{\Omega_\varepsilon, r}(u, 2\kappa) \cap \mathcal{N}_{\Omega_\varepsilon, R_0, \infty}(Z)$, we have

$$I_k \leq Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon, r}(u, 2\kappa) \cap \mathcal{N}_{\Omega_\varepsilon, R_0, \infty}(Z)) \omega(m)^{|S_{R_0+kR} \setminus S_{R_0+(k-1/2)R}|} 2^m |S_{R_0+kR} \setminus S_{R_0+(k-1)R}| \times \\ \times \left(2c^{-\frac{m}{p}} c(p, m)\right)^{|S_{R_0+(k-1/2)R}|}. \quad (\text{A.33})$$

Here, the last factor arises as the bound on the integral

$$\int_{\mathcal{N}_{S_{R_0+(k-1/2)R}, r}(u, \kappa)} \exp\left\{-c \sum_{i \in S_{R_0+(k-1/2)R}} |\nabla X(i)|^p\right\} \prod_{i \in S_{R_0+(k-1/2)R}} dX(i) \quad (\text{A.34})$$

according to Lemma A.1 a) with

$$\vartheta |S_{R_0+(k-1/2)R}|^{1+m/d} \left(c^{-\frac{m}{p}} c(p, m)\right)^{|S_{R_0+(k-1/2)R}|^{-1}} \leq \left(2c^{-\frac{m}{p}} c(p, m)\right)^{|S_{R_0+(k-1/2)R}|} \quad (\text{A.35})$$

valid for ε sufficiently small (estimating $|S_{R_0+(k-1/2)R}| \geq |S_{R_0+\frac{1}{2}R}| \sim \varepsilon^{-d+1} R = \frac{\eta}{N} \varepsilon^{-d}$).

Combining (A.29) with (A.33) for each of $N-1$ integrals over $\widehat{\mathcal{N}}_r^{(\ell)}(u, \kappa) \setminus \mathcal{M}_K$, we get

$$Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon, r}(u, \kappa) \setminus \mathcal{M}_K) \leq \\ \leq \exp\left\{\mathcal{C}\left(\left(\frac{K}{N} + \eta + \left(\frac{N\kappa}{\eta}\right)^r\right)\varepsilon^{-d} + \sum_{\substack{j \in S_{NR} \\ \tau_j(A) \subset \Omega_\varepsilon}} U(Z_{\tau_j(A)})\right)\right\} Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon, r}(u, 2\kappa) \cap \mathcal{N}_{\Omega_\varepsilon, R_0, \infty}(Z)). \quad (\text{A.36})$$

Here we bounded the prefactor $N-1$ (the number of terms with $k = 1, \dots, N-1$) combined with the factors in (A.33) by

$$N(\omega(m) 2^{m+1} c^{-\frac{m}{p}} c(p, m))^{|S_{NR}|} \leq e^{\frac{2}{3} \mathcal{C} \eta \varepsilon^{-d}} \quad (\text{A.37})$$

with a constant $\mathcal{C} = 3 \max(\tilde{C}, C_\partial |\partial\Omega| \log(\omega(m) 2^{m+1} c^{-\frac{m}{p}} c(p, m)))$. We used the bound $|S_{NR}| \leq C_\partial |\partial\Omega| \varepsilon^{-d+1} NR = C_\partial |\partial\Omega| \varepsilon^{-d} \eta$ and bounded, for ε sufficiently small, the term $N = e^{(\varepsilon^d \log N) \varepsilon^{-d}}$ by $\exp\{\frac{1}{3} \mathcal{C} \eta \varepsilon^{-d}\}$.

According to Lemma 2.1, we have $Z_{\Omega_\varepsilon}(\mathcal{M}_K \cap \mathcal{N}_{\Omega_\varepsilon, r}(u, \kappa)) \leq e^{-\frac{1}{2} K |\Omega_\varepsilon|} D^{|\Omega_\varepsilon|}$. Hence,

$$Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon, r}(u, \kappa) \cap \mathcal{M}_K) \leq \frac{1}{2} Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon, r}(u, \kappa)) \quad (\text{A.38})$$

once we choose $K = \log D + \varepsilon^d \frac{\log 2}{|\Omega|} + F_{\kappa, \varepsilon}(u)$. Then, multiplying the right hand side in (A.36) by 2, we can replace $Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon, r}(u, \kappa) \setminus \mathcal{M}_K)$ by $Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon, r}(u, \kappa))$, yielding the claim with a slight increase of \mathcal{C} and with $b = 1 + \log D$ for sufficiently small ε . \square

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